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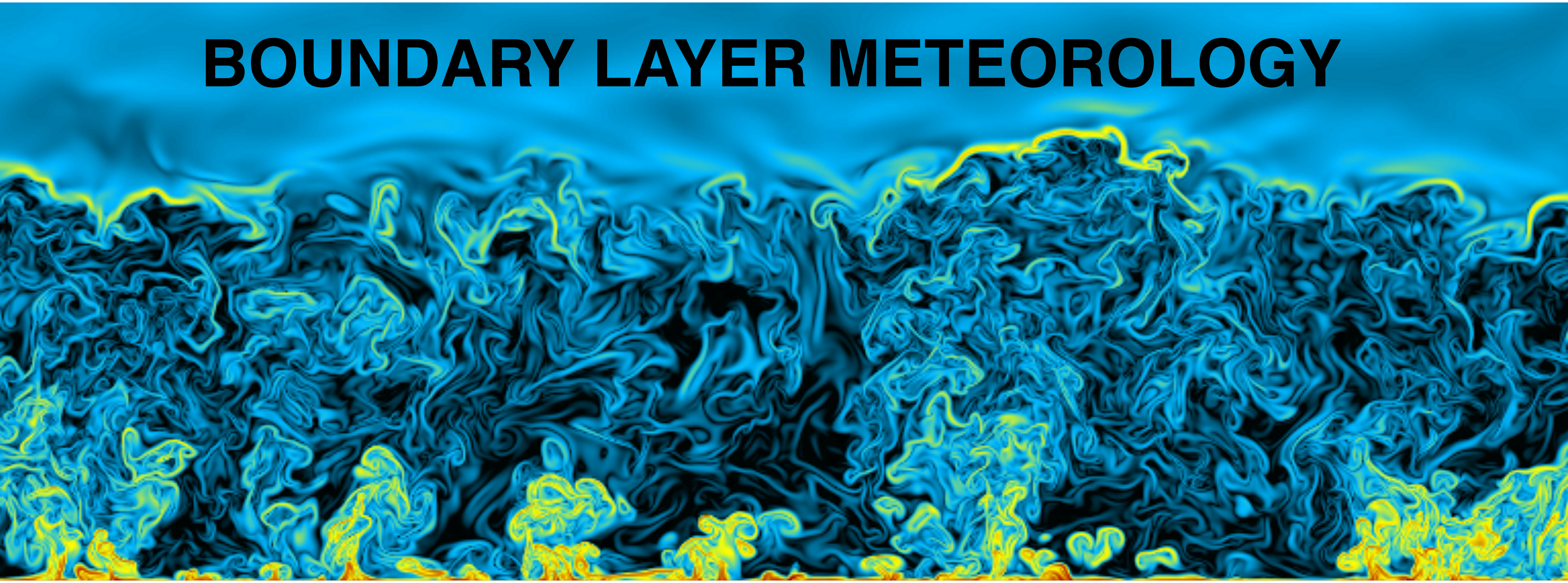

BOUNDARY LAYER METEOROLOGY

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Chapter 6

Turbulence kinetic energy (TKE)

& Dynamic Stability

Content

Chapter 6 (in script)

- TKE equation
- Stability Measures

Extra:

- TPE equation

Turbulence Kinetic Energy

- important variable characterizing turbulence (and hence PBL state)
- Recall: ‘1.5 order closure’
  → do not include *all* concervation equations for higher-order moments; but that for TKE..
- TKE is sum of velocity variances

Turbulence Kinetic Energy

Remember

→ conservation equation for higher moments:

$$\frac{\partial \overline{u_i'^2}}{\partial t} + \overline{u}_j \frac{\partial \overline{u_i'^2}}{\partial x_j} = -2\overline{u_i' u_j'} \frac{\partial \overline{u}_i}{\partial x_j} - \frac{\partial \overline{u_j' u_i'^2}}{\partial x_j}$$

$$+2\delta_{i3}\overline{u_i'\left(\frac{\theta'}{\overline{\theta}}\right)}g + 2f_c\varepsilon_{ij3}\overline{u_i' u_j'} - \frac{2}{\overline{\rho}}\overline{u'_i \frac{\partial p'}{\partial x_i}} + 2\nu \overline{u'_i \frac{\partial^2 u'_i}{\partial x_j^2}}$$

→ summed!

$$TKE = \frac{1}{2}\rho\overline{u_{ii}'^2}, \; \overline{e} = TKE / \rho$$

→ conservation equation for TKE

TKE Equation

→ simplifications
→ terms
→ interpretation

TKE Equation: Simplifications

$$\frac{\partial \overline{u'^2_i}}{\partial t} + \overline{u}_j \frac{\partial \overline{u'^2_i}}{\partial x_j} = -2\overline{u_i' u_j'}\frac{\partial \overline{u}_i}{\partial x_j} - \frac{\partial \overline{u_j' u'^2_i}}{\partial x_j}$$

$$+2\delta_{i3}\overline{u_i'\left(\frac{\theta'}{\overline{\theta}}\right)}g + 2f_c\varepsilon_{ij3}\overline{u_i'u_j'} - \underline{\frac{2}{\overline{\rho}}\overline{u'_i\frac{\partial p'}{\partial x_i}}} + 2\nu\overline{u'_i\frac{\partial^2 u'_i}{\partial x_j^2}}$$

1) pressure term:

= 0 (continuity equation)

holds: $$\frac{\partial(u'_i\, p')}{\partial x_i} = u'_i\frac{\partial p'}{\partial x_i} + p'\cancel{\frac{\partial u'_i}{\partial x_i}}$$

$$\longrightarrow \quad -\frac{2}{\overline{\rho}}\overline{u'_i\frac{\partial p'}{\partial x_i}} = -\frac{2}{\overline{\rho}}\frac{\partial \overline{u'_i\, p'}}{\partial x_i}$$

‘pressure transport term’

TKE Equation: Simplifications

$$\frac{\partial \overline{u'^2_i}}{\partial t} + \overline{u}_j \frac{\partial \overline{u'^2_i}}{\partial x_j} = -2\overline{u'_i u'_j}\frac{\partial \overline{u}_i}{\partial x_j} - \frac{\partial \overline{u'_j u'^2_i}}{\partial x_j}$$

$$+2\delta_{i3}\overline{u'_i\left(\frac{\theta'}{\overline{\theta}}\right)}g + 2f_c\varepsilon_{ij3}\overline{u'_i u'_j} - \frac{2}{\overline{\rho}}\overline{u'_i \frac{\partial p'}{\partial x_i}} + \underline{2\nu \overline{u'_i \frac{\partial^2 u'_i}{\partial x_j^2}}}$$

2) dissipation:

holds: $$\frac{\partial^2 u'^2_i}{\partial x_j^2} = \frac{\partial}{\partial x_j}\left(2u'_i \frac{\partial u'_i}{\partial x_j}\right) = 2\left(\frac{\partial u'_i}{\partial x_j}\right)^2 + 2u'_i\frac{\partial^2 u'_i}{\partial x_j^2}$$

therefore: $$\underline{2\nu u'_i \frac{\partial^2 u'_i}{\partial x_j^2}} = \nu\frac{\partial^2 u'^2_i}{\partial x_j^2} - 2\nu\left(\frac{\partial u'_i}{\partial x_j}\right)^2$$

TKE Equation: Simplifications

therefore:

$$\underline{2\nu u'_i \frac{\partial^2 u'_i}{\partial x_j^2}} = \underbrace{\nu \frac{\partial^2 u'^2_i}{\partial x_j^2}}_{O(10^{-10})} - \underbrace{2\nu\left(\frac{\partial u'_i}{\partial x_j}\right)^2}_{O(10^{-3})}$$

→ Def:

$$\boxed{\varepsilon =: \nu\overline{\left(\frac{\partial u'_i}{\partial x_j}\right)^2}}$$

rate of dissipation of TKE

→ dissipation: conversion of TKE into heat

always: loss term

always negative!

Levi-Civita Symbol

$$\boxed{\varepsilon_{ijk}}$$

Permutation symbol:

→ = 1 if an even number of permutations is required to obtain an increasing sequence (1,2,3), (2,3,1), (3,1,2) [after max you restart with 1]

→ = - 1 if an odd number of permutations is required to obtain an increasing sequence (3,2,1), (2,1,3), (1,3,2)

→ = 0, otherwise (two indices have the same value)

TKE Equation: Simplifications

$$\frac{\partial \overline{u'^2_i}}{\partial t} + \bar{u}_j \frac{\partial \overline{u'^2_i}}{\partial x_j} = -2\overline{u'_i u'_j}\frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \overline{u'_j u'^2_i}}{\partial x_j}$$

$$+2\delta_{i3}\overline{u'_i\left(\frac{\theta'}{\bar{\theta}}\right)}g + \cancel{2f_c\varepsilon_{ij3}\overline{u'_i u'_j}} - \frac{2}{\bar{\rho}}\overline{u'_i\frac{\partial p'}{\partial x_i}} + 2\nu\overline{u'_i\frac{\partial^2 u'_i}{\partial x_j^2}}$$

3) Coriolis term:

holds: $2f_c\varepsilon_{ij3}\overline{u'_i u'_j} = 2f_c\varepsilon_{213}\overline{u'_2 u'_1}$

$$+ 2f_c\varepsilon_{123}\overline{u'_1 u'_2}$$

$$= -2f_c\overline{u'_2 u'_1} + 2f_c\overline{u'_1 u'_2}$$

$$= 0$$

→ all other $e_{ij3}=0$!

TKE Equation: Simplifications

$$\frac{\partial \overline{u'^2_i}}{\partial t} + \overline{u}_j \frac{\partial \overline{u'^2_i}}{\partial x_j} = -2\overline{u_i' u_j'}\frac{\partial \overline{u}_i}{\partial x_j} - \frac{\partial \overline{u_j' u'^2_i}}{\partial x_j}$$

$$+2\delta_{i3}\overline{u_i'\left(\frac{\theta'}{\overline{\theta}}\right)}g + \underline{\cancel{2f_c\varepsilon_{ij3}\overline{u_i' u_j'}}} - \frac{2}{\overline{\rho}}\overline{u'_i\frac{\partial p'}{\partial x_i}} + 2\nu\overline{u'_\mathrm{i}\frac{\partial^2 u'_i}{\partial x_j^2}}$$

3) Coriolis term:

holds:

$$2f_c\varepsilon_{ij3}\overline{u'_\mathrm{i}\,u'_\mathrm{j}} = 2f_c\varepsilon_{213}\overline{u'_2\,u'_1}$$

$$+\,2f_c\varepsilon_{123}\overline{u'_1\,u'_2}$$

$$= -2f_c\overline{u'_2\,u'_1} + 2f_c\overline{u'_1\,u'_2}$$

$$= 0$$

→ all other

e_{ij3}=0!

TKE Equation

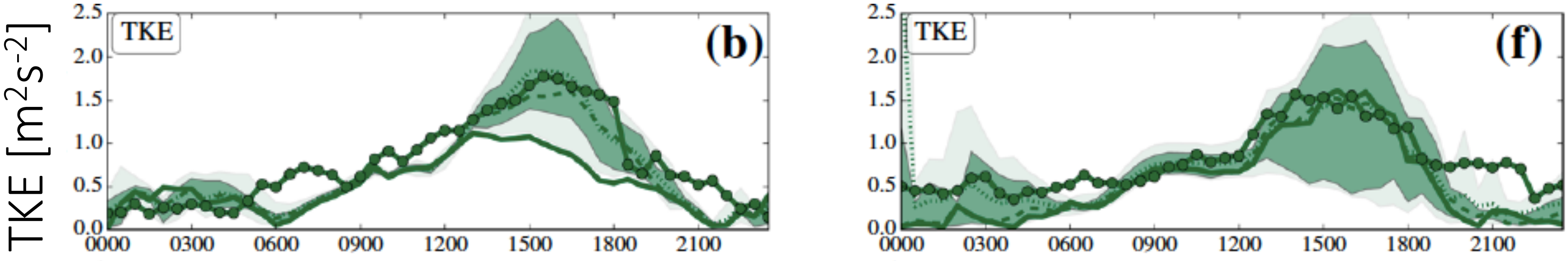
$$\frac{\partial \overline{e}}{\partial t}+\overline{u}_j\frac{\partial \overline{e}}{\partial x_j}=-\overline{u_i'u_j'}\frac{\partial \overline{u}_i}{\partial x_j}-\frac{\partial \overline{u_j'e}}{\partial x_j}+\delta_{i3}\overline{u_i'\theta'}\frac{g}{\overline{\theta}}-\frac{1}{\overline{\rho}}\frac{\partial \overline{u_j'p'}}{\partial x_j}-\varepsilon$$

I II III IV V VI VII

I: local temporal change: daily cycle!

TKE Equation: Daily Cycle

two sites in the Inn Valley



Goger et al 2018

line: COSMO (1 km) model and range
symbols : i-Box measurements

night: ‘calm’ → 0.1 – 0.5 m^{2}s^{-2}
day: → 1-10 m^{2}s^{-2} (the latter is a storm)

TKE Equation

$$\frac{\partial \bar{e}}{\partial t} + \bar{u}_j \frac{\partial \bar{e}}{\partial x_j} = -\overline{u'_i u'_j}\frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \overline{u'_j e}}{\partial x_j} + \delta_{i3}\overline{u'_i \theta'}\frac{g}{\bar{\theta}} - \frac{1}{\bar{\rho}}\frac{\partial \overline{u'_j p'}}{\partial x_j} - \varepsilon$$

I II III IV V VI VII

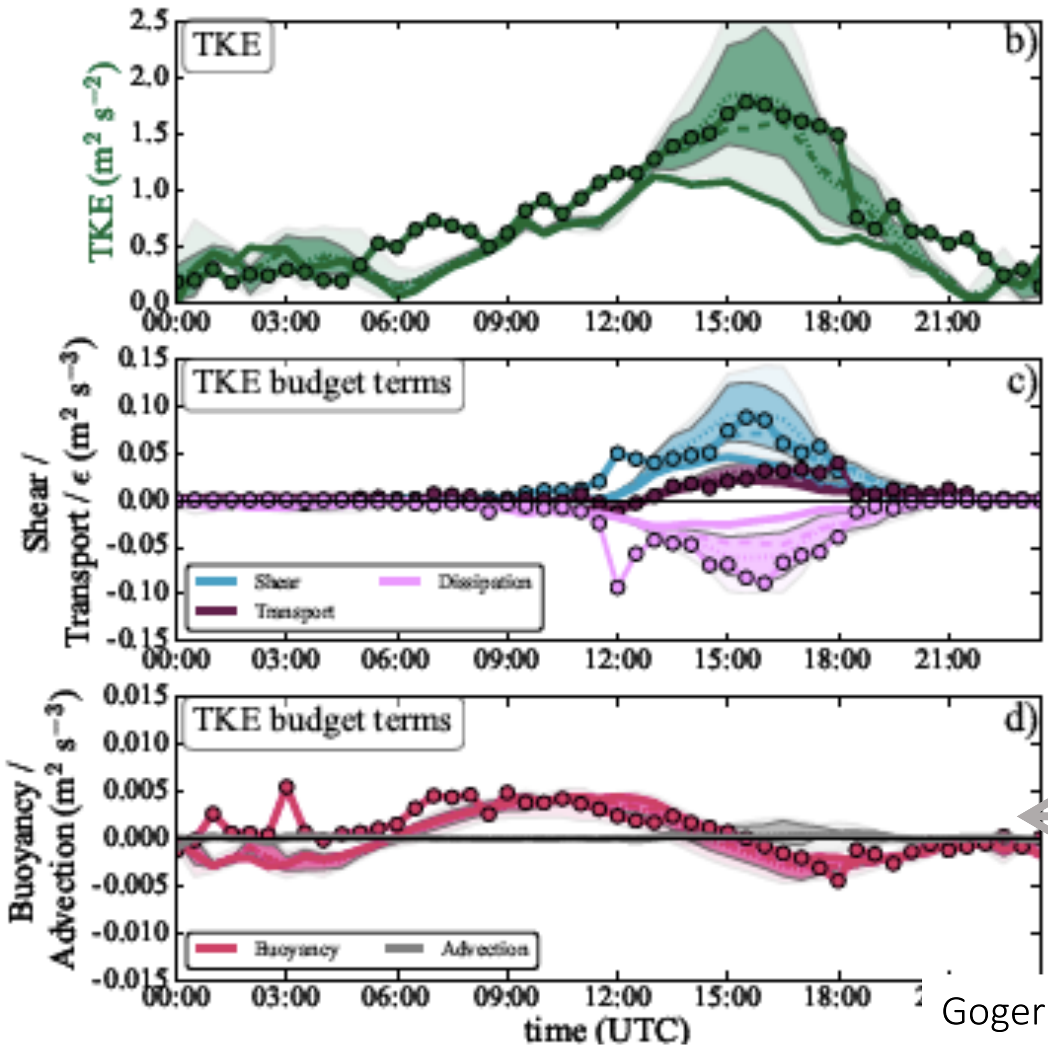
I: local temporal change: daily cycle!

II: advection → little known....

→ generally thought to be small

TKE Equation: Advection

site: ‚Kolsass'



Goger et al 2018

In horizontally homogeneous and flat (HHF) terrain → zero

→ test of NWP model (COSMO) in i-Box

→ in non-HHF terrain (Inn Valley): advection?

advection term
→ very small....

TKE Equation

$$\frac{\partial \bar{e}}{\partial t} + \bar{u}_j \frac{\partial \bar{e}}{\partial x_j} = -\overline{u'_i u'_j} \frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \overline{u'_j e}}{\partial x_j} + \delta_{i3} \overline{u'_i \theta'} \frac{g}{\bar{\theta}} - \frac{1}{\bar{\rho}} \frac{\partial \overline{u'_j p'}}{\partial x_j} - \varepsilon$$

I II III IV V VI VII

I: local temporal change: daily cycle!

II: advection

III: shear **production**

TKE Equation: Shear Production

$$\frac{\partial \bar{e}}{\partial t} + \bar{u}_j \frac{\partial \bar{e}}{\partial x_j} = \enclose{circle}{-}\overline{u'_i u'_j}\frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \overline{u'_j e}}{\partial x_j} + \delta_{i3}\overline{u'_i \theta'}\frac{g}{\bar{\theta}} - \frac{1}{\bar{\rho}}\frac{\partial \overline{u'_j p'}}{\partial x_j} - \varepsilon$$

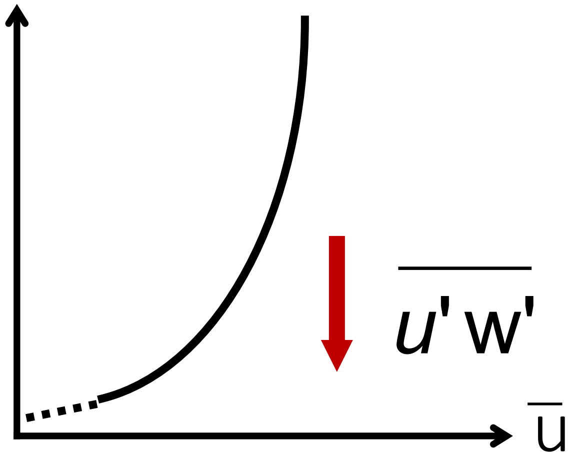
I II III IV V VI VII

$$\frac{\partial \bar{u}}{\partial z} > 0, \ \overline{u'w'} < 0$$

other ij analogous:

→ characteristic gradient

→ deformation (flux)



TKE Equation: Shear production

→ ‘always’ positive!

→ large when ‘large wind’ , small when less wind

→ **interaction** of turbulence with mean flow

TKE: $$\frac{\partial \overline{e}}{\partial t} + \ldots = \boxed{-}\,\overline{u'_i\, u'_j}\,\frac{\partial \overline{u}_i}{\partial x_j} + \ldots..$$

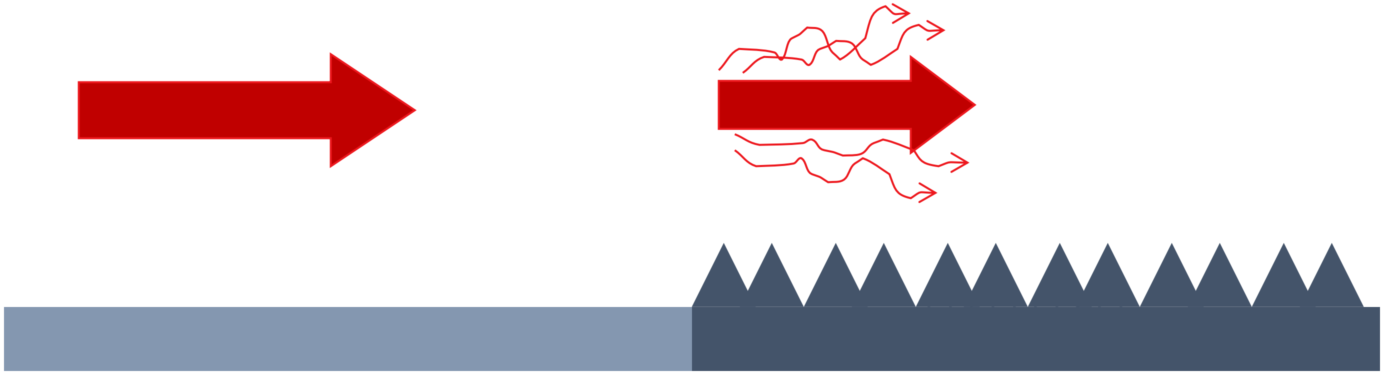
MKE: $$\frac{\partial E}{\partial t} + \ldots = \boxed{+}\,\overline{u'_i\, u'_j}\,\frac{\partial \overline{u}_i}{\partial x_j} + \ldots..$$

TKE Equation: Shear production

→ **interaction** of turbulence with mean flow

→ if mean flow becomes (more) turbulent, it looses mean kinetic energy ....

→ .... and gains TKE



→ 'always' production term

→ in ideal conditions and neutral stratification: *the* production term

TKE Equation

$$\frac{\partial \bar{e}}{\partial t} + \bar{u}_j \frac{\partial \bar{e}}{\partial x_j} = -\overline{u'_i u'_j} \frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \overline{u'_j e}}{\partial x_j} + \delta_{i3} \overline{u'_i \theta'} \frac{g}{\bar{\theta}} - \frac{1}{\bar{\rho}} \frac{\partial \overline{u'_j p'}}{\partial x_j} - \varepsilon$$

I II III IV V VI VII

I: local temporal change: daily cycle!

II: advection

III: shear production

IV: turbulent transport of TKE (its divergence!)

TKE Equation: Turbulent transport

$$-\frac{\partial \overline{u_j' e}}{\partial x_j}$$

→ not magnitude but divergence!

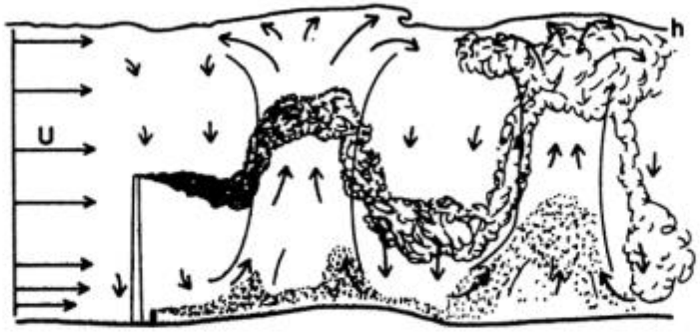
→ horiz. homogeneous: esp. vertical transport

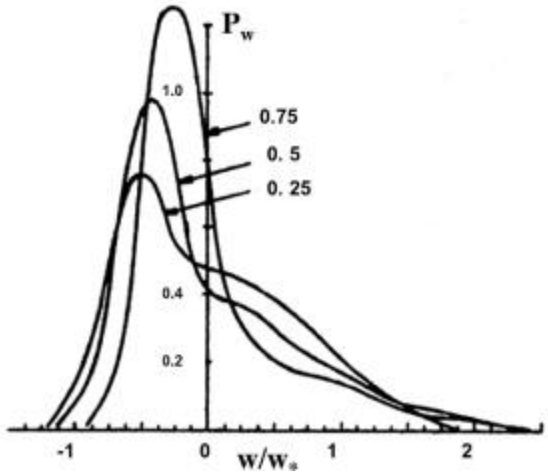
$$-\frac{\partial \overline{u_3' e}}{\partial x_3} = -\frac{\partial \overline{w' e}}{\partial z}$$

$$\partial \overline{w'e} \big/ \partial z = \partial \overline{(w'u_1'^2 + w'u_2'^2 + w'u_3'^2)} \big/ \partial z$$

$$\partial \overline{w'e} \big/ \partial z = \partial \overline{(w'u'^2 + w'v'^2 + \boxed{w'^3})} \big/ \partial z$$

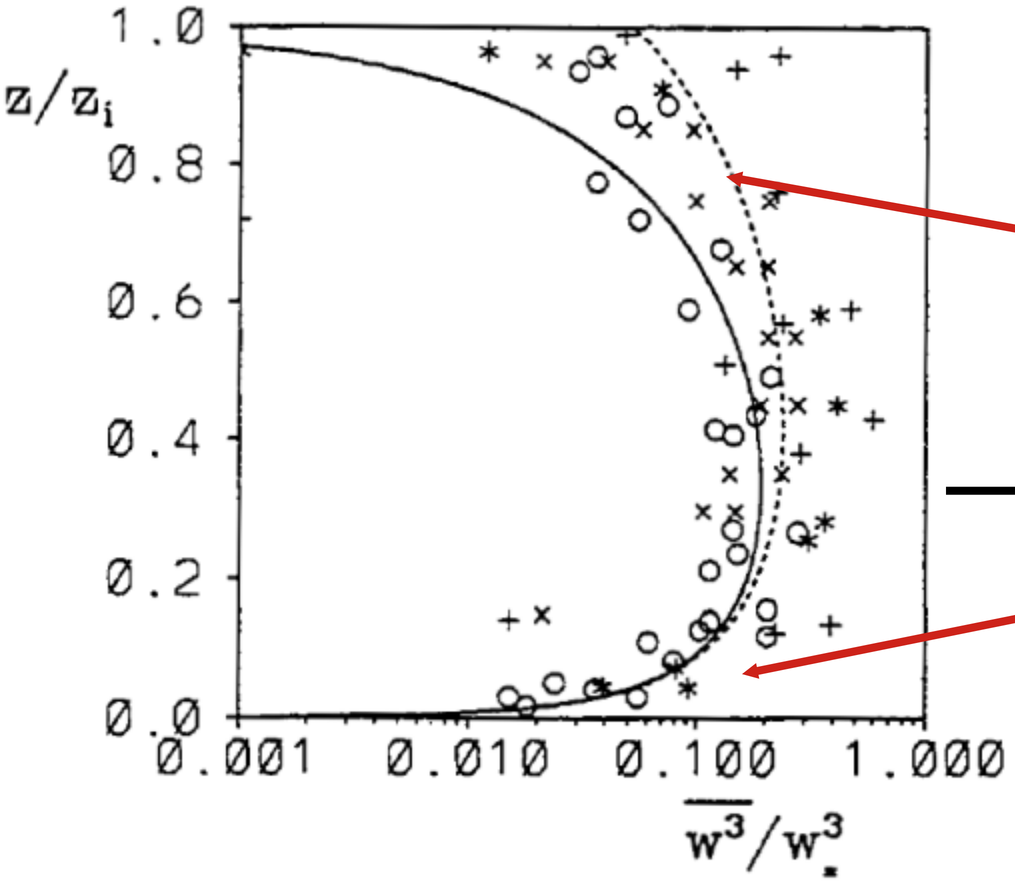
→ CBL, in particular





TKE Equation: Turbulent transport

$$-\partial \overline{w'e}/\partial z = -\partial \overline{(w'u'^2 + w'v'^2 + w'^3)}/\partial z$$



gain term

$$\frac{\partial \overline{w'^3}}{\partial z} < 0$$

IMPORT

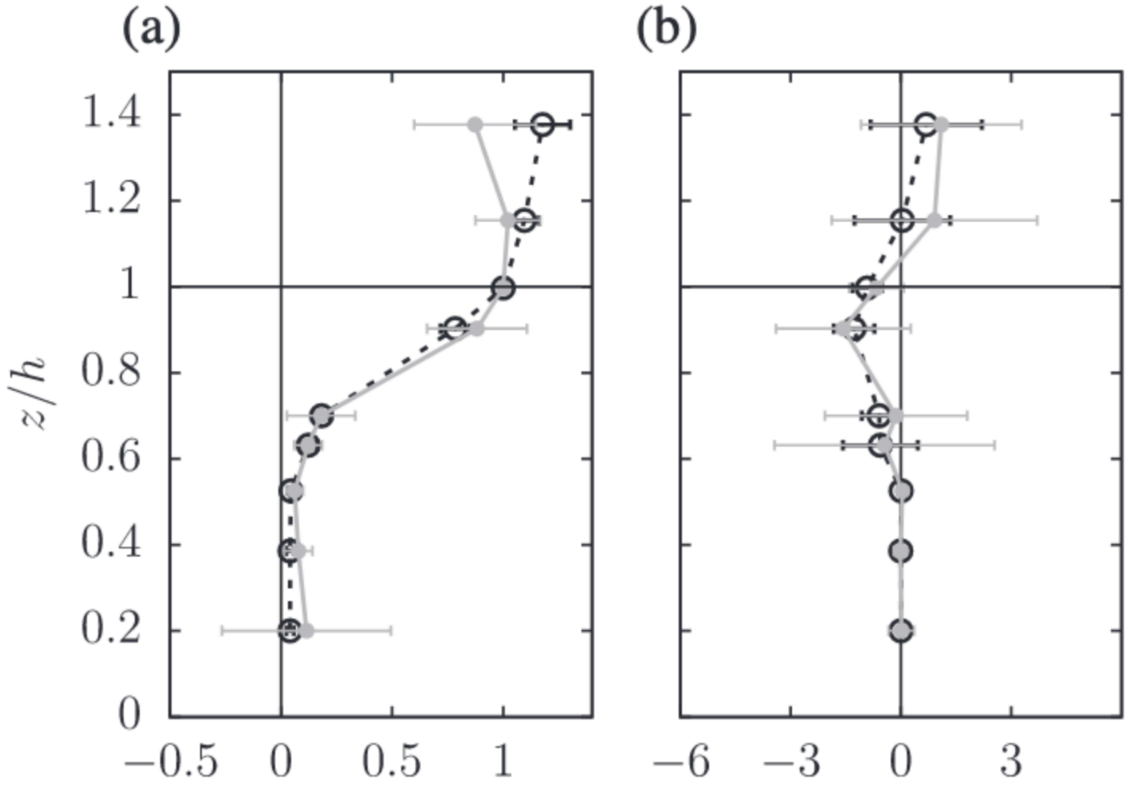
production:
→ EXPORT

$$\frac{\partial \overline{w'^3}}{\partial z} > 0$$

loss term
(negative sign in eq.)

TKE Equation: Turbulent transport

Especially important in canopy flows



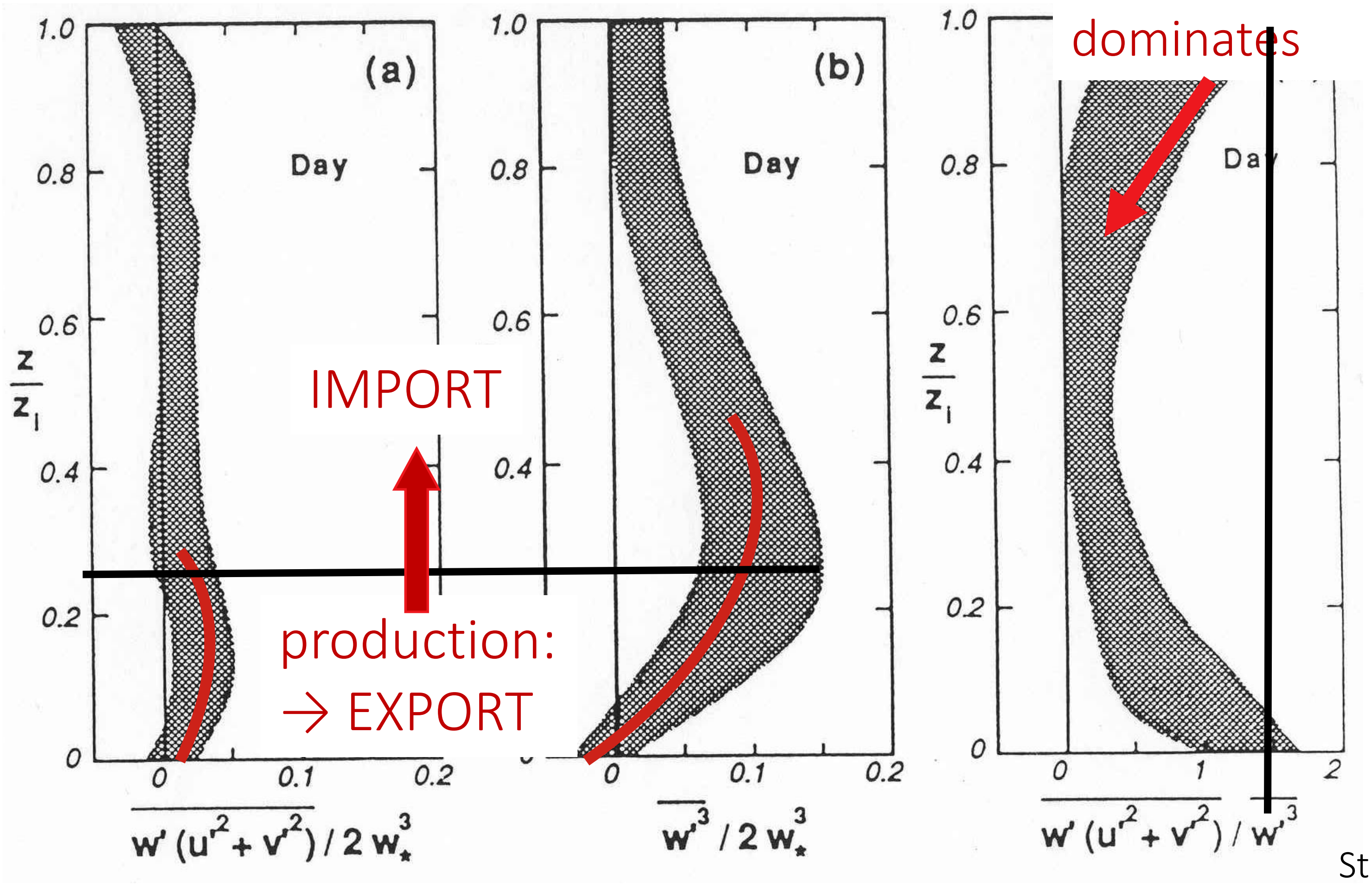
Normalized TKE

Normalized TKE transport

Freire et al. 2019

TKE Equation: Turbulent transport

Also, but less pronounced: horizontal components



Vertical contribution dominates

IMPORT

production: → EXPORT

Stull (1988)

TKE Equation

$$\frac{\partial \overline{e}}{\partial t} + \overline{u}_j \frac{\partial \overline{e}}{\partial x_j} = -\overline{u'_i u'_j} \frac{\partial \overline{u}_i}{\partial x_j} - \frac{\partial \overline{u'_j e}}{\partial x_j} + \delta_{i3} \overline{u'_i \theta'} \frac{g}{\overline{\theta}} - \frac{1}{\overline{\rho}} \frac{\partial \overline{u'_j p'}}{\partial x_j} - \varepsilon$$

I II III IV V VI VII

I: local temporal change: daily cycle!

II: advection

III: shear production

IV: turbulent transport of TKE

V: buoyancy term

TKE Equation: Buoyancy

$$\frac{\partial \bar{e}}{\partial t} + \bar{u}_j \frac{\partial \bar{e}}{\partial x_j} = -\overline{u'_i u'_j}\frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \overline{u'_j e}}{\partial x_j} + \boxed{\delta_{i3}\overline{u'_i \theta'}\frac{g}{\bar{\theta}}} - \frac{1}{\bar{\rho}}\frac{\partial \overline{u'_j p'}}{\partial x_j} - \varepsilon$$

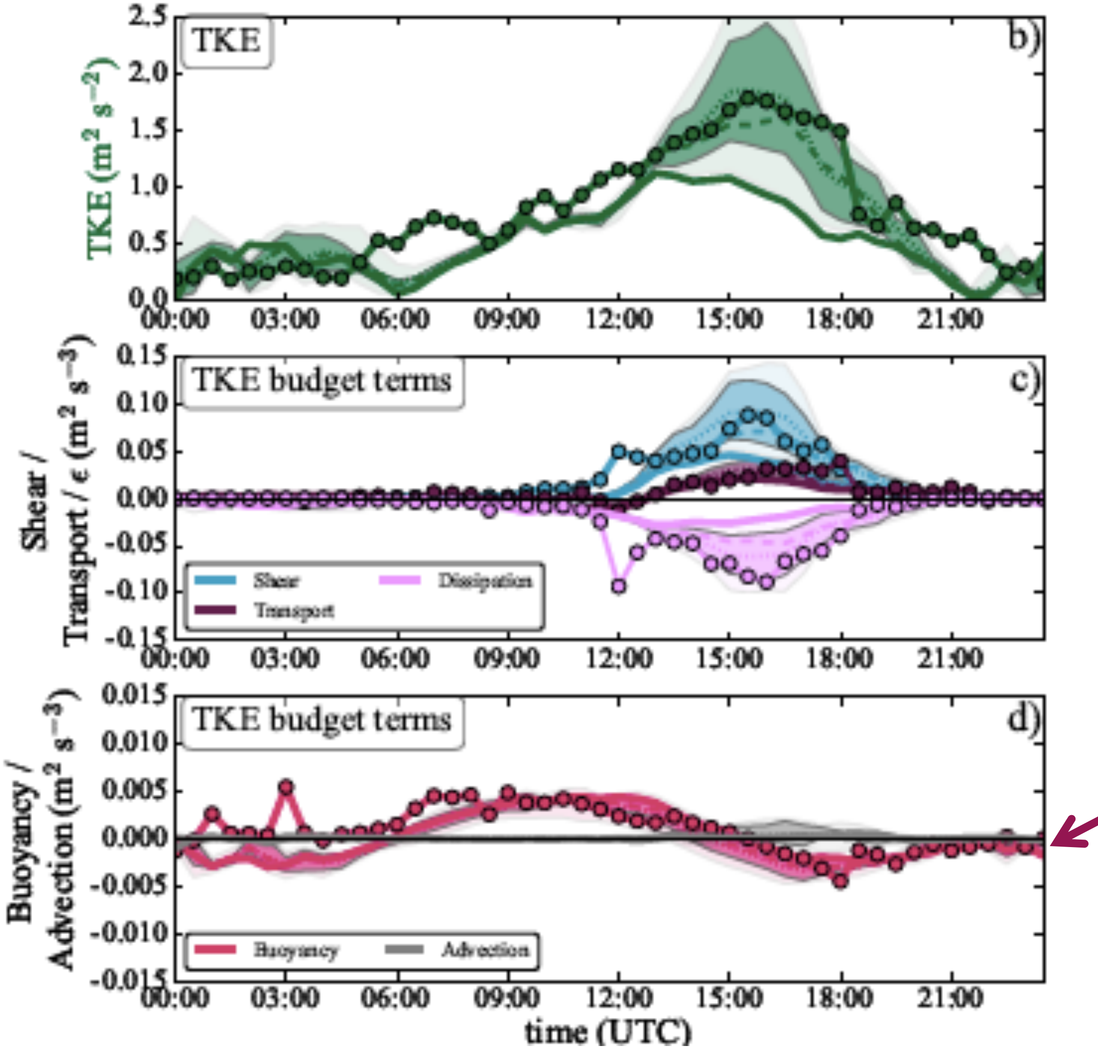
→ only vertical component (δ_{i3})

→ production *or* damping

→ sign of $\overline{w'\theta'}$

→ Boussinesq approximation: ‘now visible’!

TKE Equation: Buoyancy



site: ‚Kolsass‘

Goger et al 2018

TKE Equation

$$\frac{\partial \overline{e}}{\partial t} + \overline{u}_j \frac{\partial \overline{e}}{\partial x_j} = -\overline{u'_i u'_j} \frac{\partial \overline{u}_i}{\partial x_j} - \frac{\partial \overline{u'_j e}}{\partial x_j} + \delta_{i3} \overline{u'_i \theta'} \frac{g}{\overline{\theta}} - \frac{1}{\overline{\rho}} \frac{\partial \overline{u'_j p'}}{\partial x_j} - \varepsilon$$

I II III IV V VI VII

I: local temporal change: daily cycle!

II: advection

III: shear production

IV: turbulent transport of TKE

V: buoyancy term

VI: pressure transport term

TKE Equation: Pressure redistribution

$$\frac{\partial \bar{e}}{\partial t} + \bar{u}_j \frac{\partial \bar{e}}{\partial x_j} = -\overline{u'_i u'_j}\,\frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \overline{u'_j e}}{\partial x_j} + \delta_{i3}\,\overline{u'_i \theta'}\,\frac{g}{\bar{\theta}} - \boxed{\frac{1}{\bar{\rho}}\,\frac{\partial \overline{u'_j p'}}{\partial x_j}} - \varepsilon$$

→ “Return to Isotropy” term: exchanges energy between u, v and w components

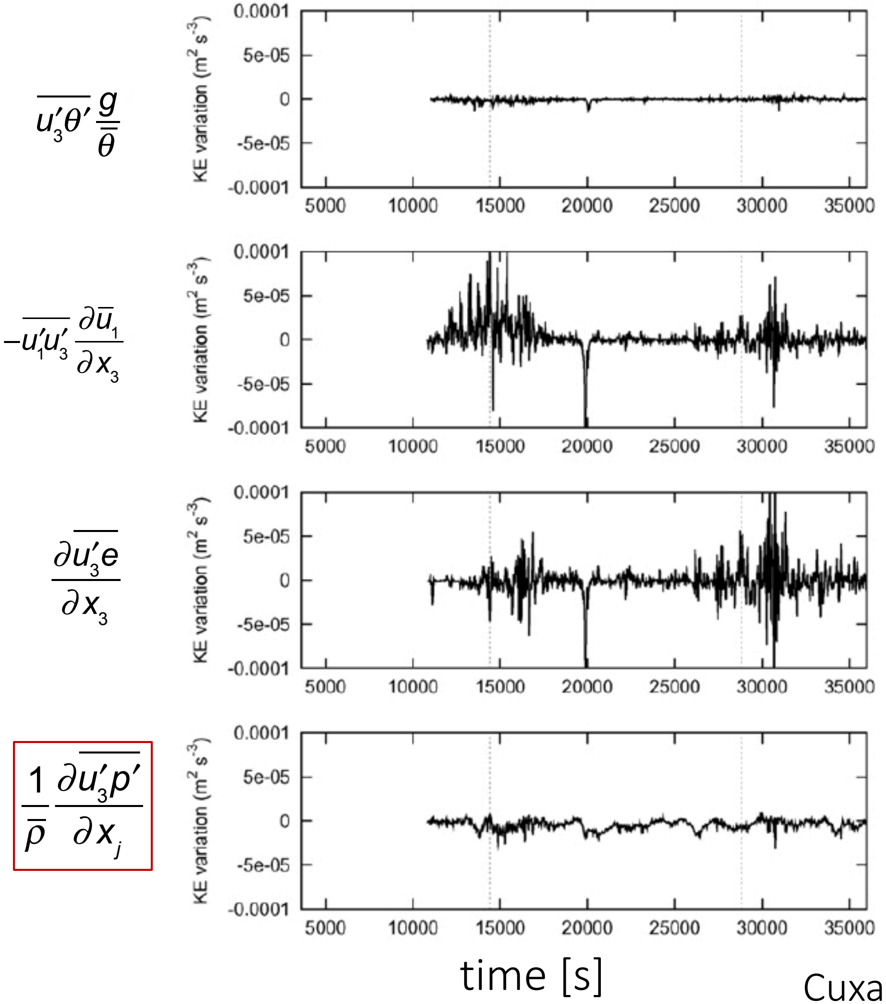
→ p’ very small normally

→ difficult to measure

→ often determined as residual

→ in numerical models: often parameterized together with TKE transport term ….

TKE Equation: Pressure redistribution



Cuxart et al. (2002)

example: CASES ’99

(layer: 1.5-30 m)

→ stable night

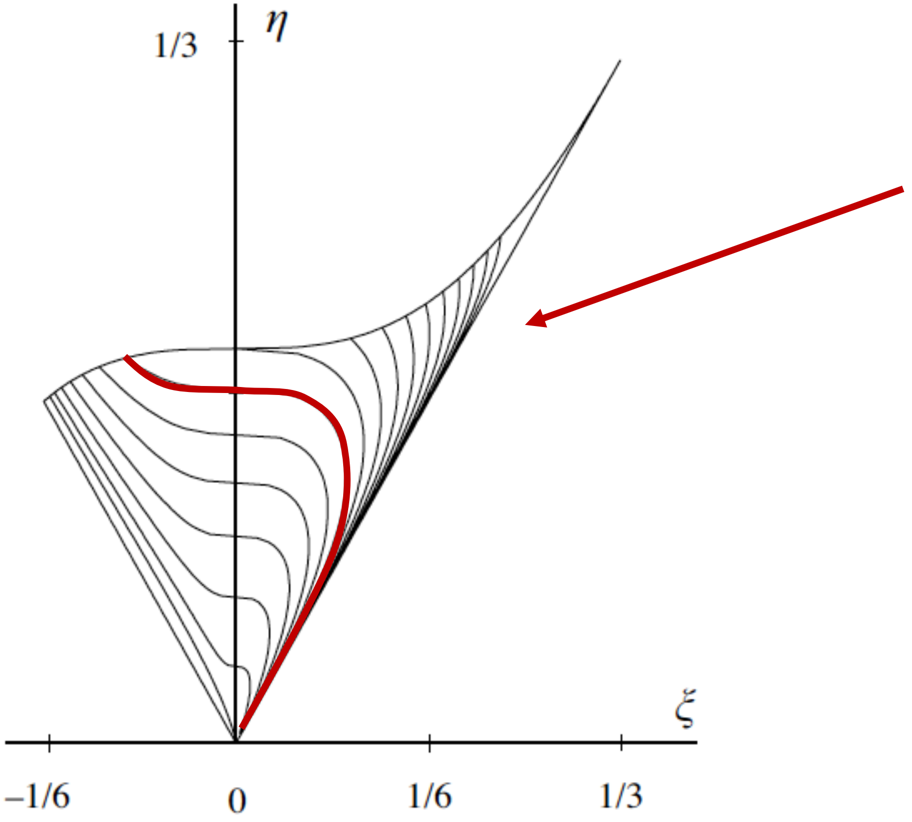
→ pressure transport: similar *magnitude*

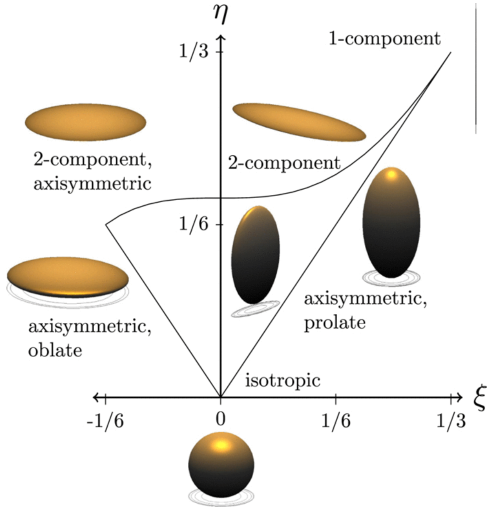
→ ‘not correlated’ with TKE transport

→ parameterization maybe not optimal...

TKE Equation: Pressure redistribution

→ Alternative: Parametrize through Return to isotropy trajectories





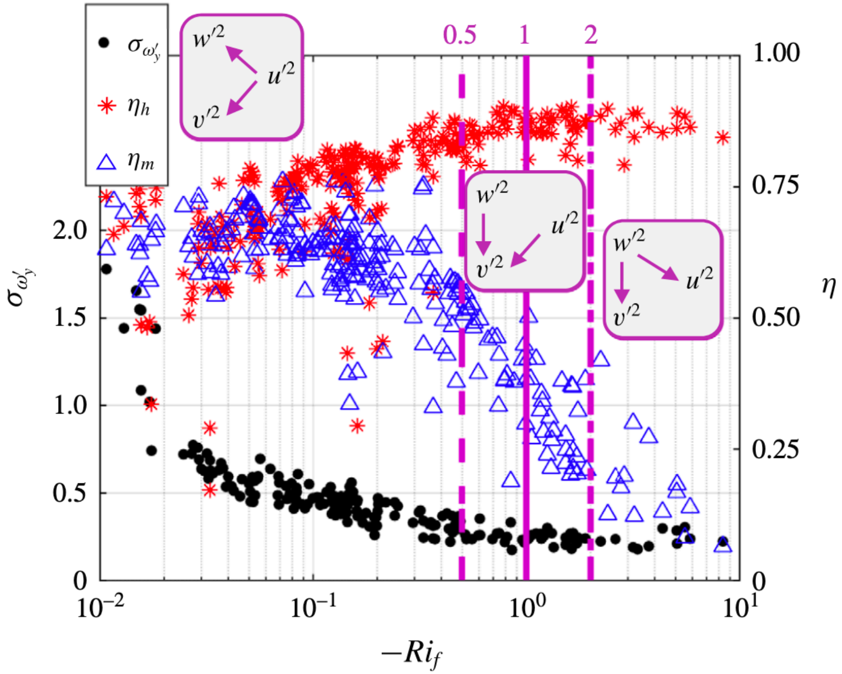
Pope (2000)

TKE Equation: Pressure redistribution

- Pressure redistribution in unstable conditions:
  Depends on the TKE source

Shear driven turbulence
→ energy in u component
→ Redistributed to v and w

Buoyancy driven turbulence
(free convection)
→ energy in w component
→ Redistributed to u and v



Bou-Zaid et al. (2018)

TKE Equation

$$\frac{\partial \overline{e}}{\partial t} + \overline{u}_j \frac{\partial \overline{e}}{\partial x_j} = -\overline{u'_i u'_j} \frac{\partial \overline{u}_i}{\partial x_j} - \frac{\partial \overline{u'_j e}}{\partial x_j} + \delta_{i3} \overline{u'_i \theta'} \frac{g}{\overline{\theta}} - \frac{1}{\overline{\rho}} \frac{\partial \overline{u'_j p'}}{\partial x_j} - \varepsilon$$

I II III IV V VI VII

I: local temporal change: daily cycle!

II: advection

III: shear production

IV: turbulent transport of TKE

V: buoyancy term

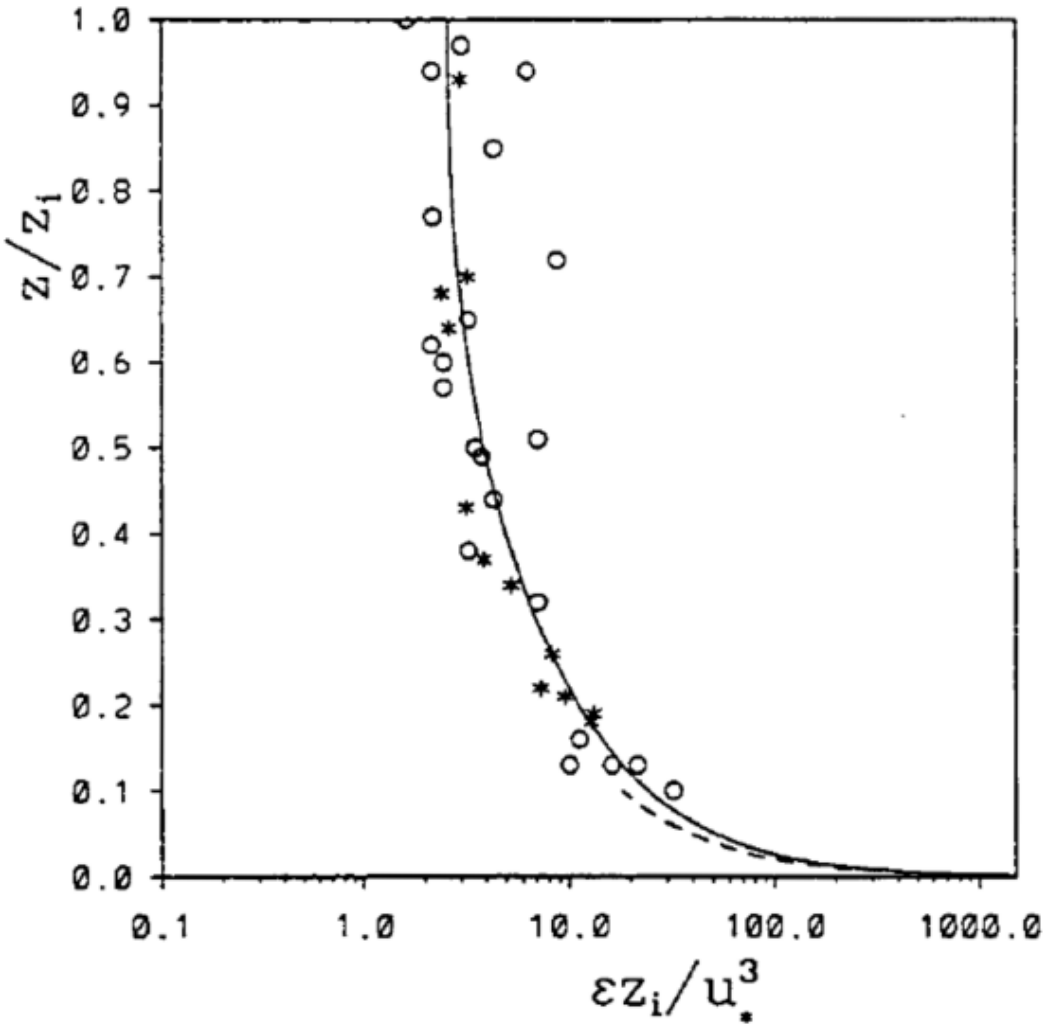
VI: pressure correlation term

VII: dissipation

TKE Equation: Dissipation

dissipation:

→ always negative!
→ large of course
at the ground



TKE Budget: Day

Stull (1988)

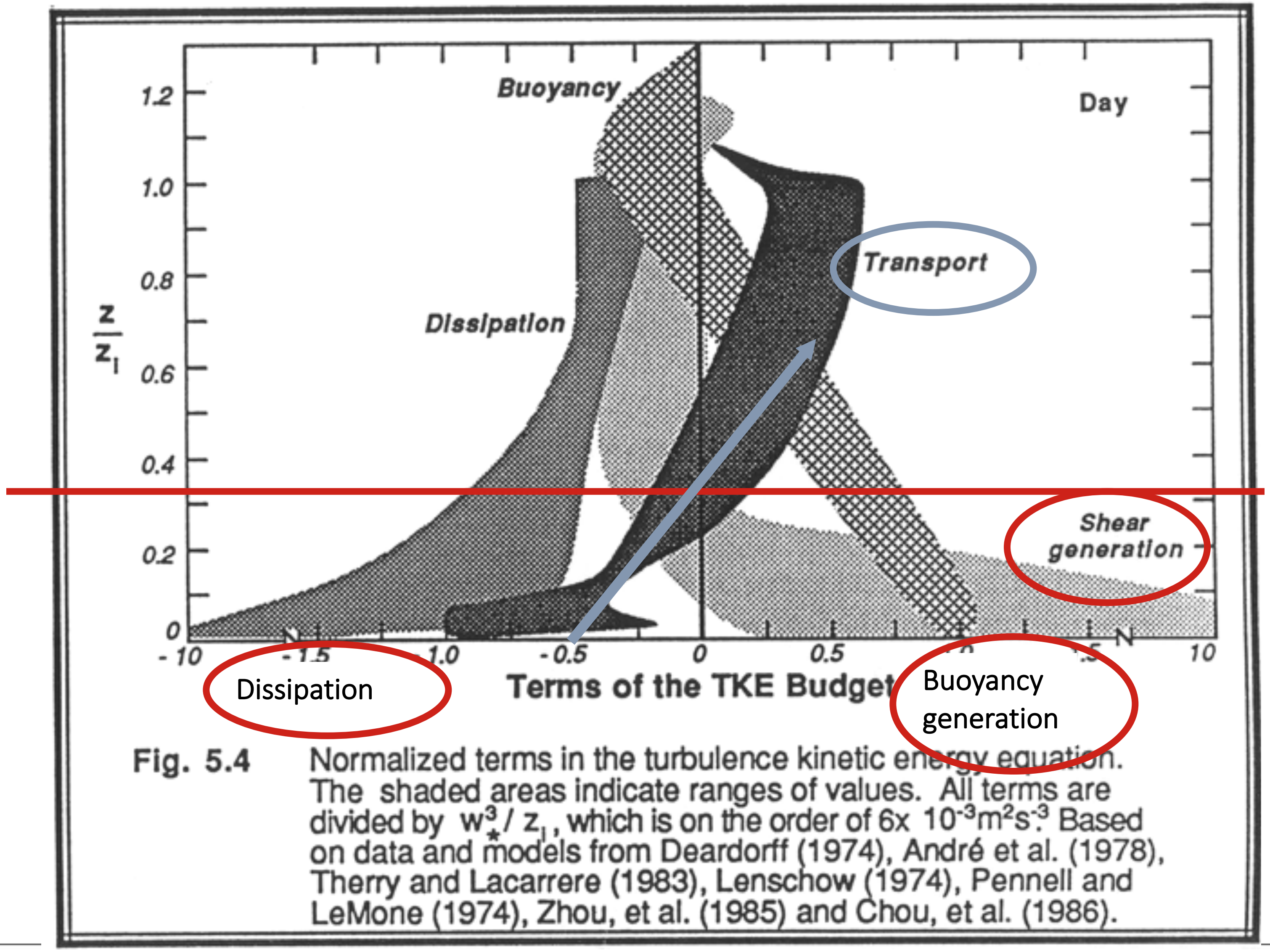


Fig. 5.4 Normalized terms in the turbulence kinetic energy equation. The shaded areas indicate ranges of values. All terms are divided by w_*^3 / z_i, which is on the order of 6x $10^{-3} m^2 s^{-3}$. Based on data and models from Deardorff (1974), André et al. (1978), Therry and Lacarrere (1983), Lenschow (1974), Pennell and LeMone (1974), Zhou, et al. (1985) and Chou, et al. (1986).

TKE Budget: Negative shear production?

Stull (1988)

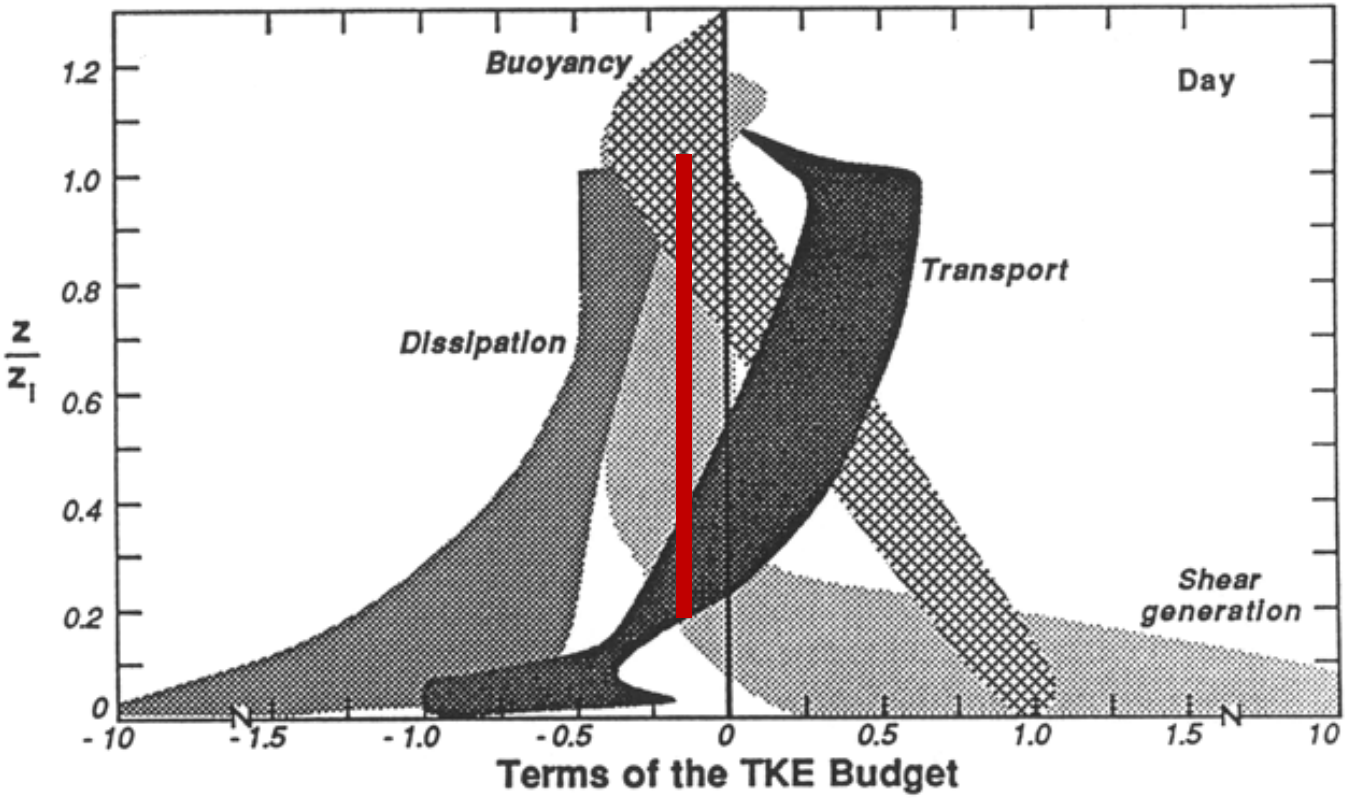


Fig. 5.4 Normalized terms in the turbulence kinetic energy equation. The shaded areas indicate ranges of values. All terms are divided by w_*^3 / z_i, which is on the order of 6x $10^{-3} m^2 s^{-3}$. Based on data and models from Deardorff (1974), André et al. (1978), Therry and Lacarrere (1983), Lenschow (1974), Pennell and LeMone (1974), Zhou, et al. (1985) and Chou, et al. (1986).

TKE Budget: Negative shear production?

Shear production term: $-\overline{u_i' u_j'} \frac{\partial \overline{u_i}}{\partial x_j}$

assume: horizontally homogeneous, $\overline{w}$=0

CBL: $\overline{u'w'} < 0$

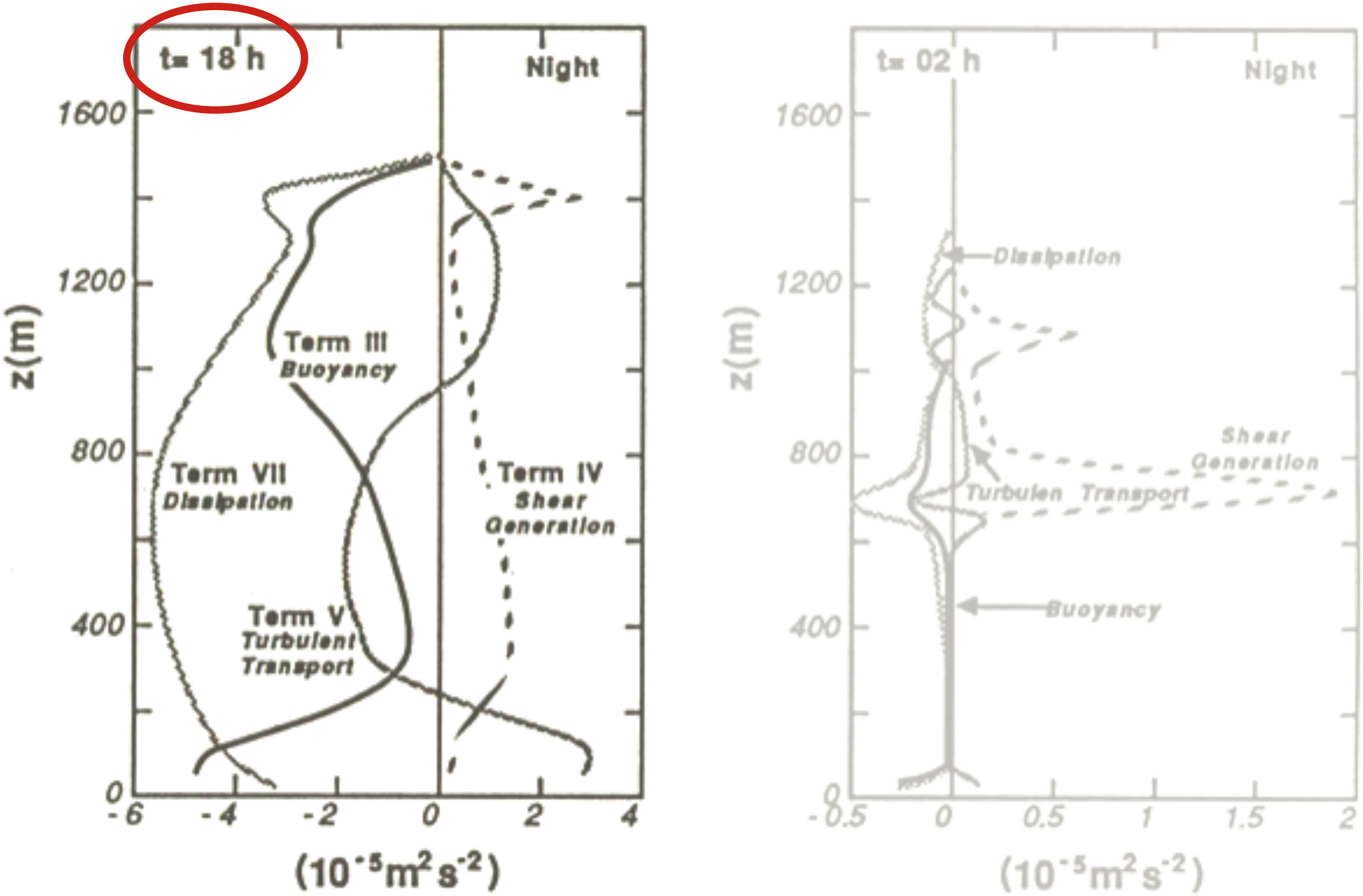
→ slightly negative $\frac{\partial \bar{u}}{\partial z}$

or: impact of directional shear?

$\overline{v'w'} \gtreqless 0$

→ slightly negative $\frac{\partial \bar{v}}{\partial z} \gtreqless 0$

TKE Budget: Night



Terms of the TKE Budget

Stull (1988)

TKE Budget

- measure for turbulence
- remember: 1.5 order closure (‘TKE closure’)
- in numerical models: description of turbulence (transport and mixing) is based on TKE (if not even simpler closure)
- most often: ‘BL approximation’
  → only vertical (horizontally homogeneous)
  → K theory as a basis

$$\frac{D\overline{e}}{Dt} = -\overline{u_1' u_3'}\frac{\partial \overline{u}_1}{\partial x_3} - \frac{\partial \overline{u_3' e}}{\partial x_3} + \overline{u_3' \theta'}\frac{g}{\overline{\overline{\theta}}} - \frac{1}{\overline{\rho}}\frac{\partial \overline{u_3' p'}}{\partial x_3} - \varepsilon$$

Example: TKE Budget Closure

Goger et al (2018):

- test TKE closure in Inn Valley
- (using COSMO-1 model)

$$\frac{D\bar{e}}{Dt} = -\overline{u_1'u_3'}\frac{\partial \bar{u}_1}{\partial x_3} - \frac{\partial \overline{u_3'e}}{\partial x_3} + \overline{u_3'\theta'}\frac{g}{\bar{\theta}} - \frac{1}{\bar{\rho}}\frac{\partial \overline{u_3'p'}}{\partial x_3} - \varepsilon$$

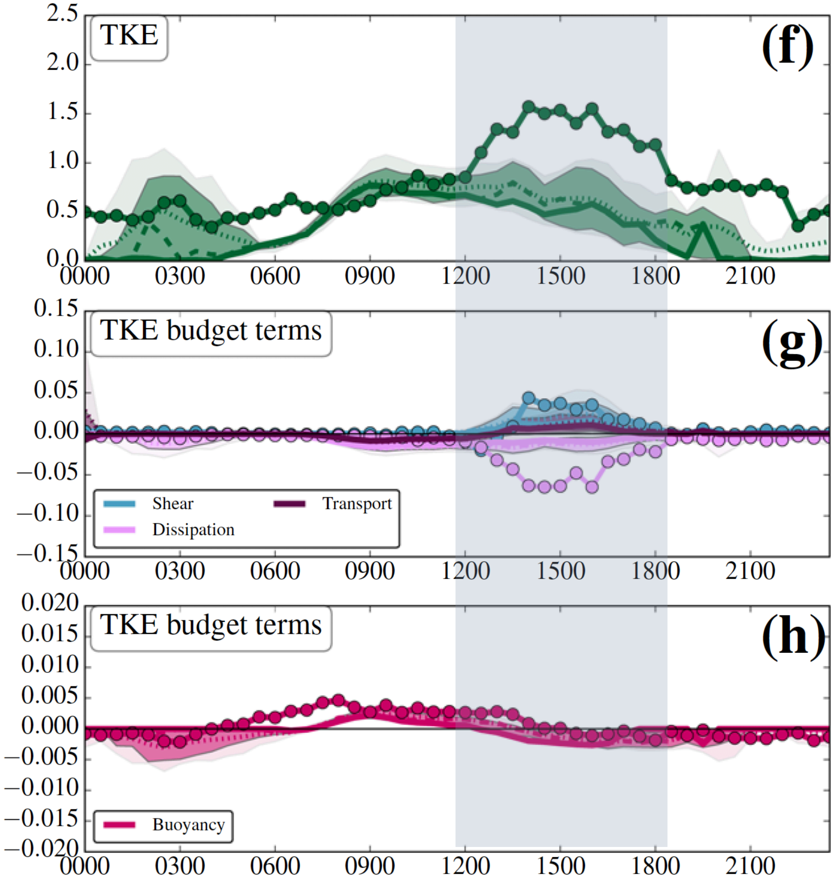
In the model:

$$\underbrace{\frac{D}{Dt}\left(\frac{q^2}{2}\right)}_{\text{tendency}} = - \underbrace{K_H \frac{g}{\theta}\frac{\partial \theta}{\partial z}}_{\substack{\text{buoyancy}\\ \text{production/consumption}}} + \underbrace{K_M\left[\left(\frac{\partial U}{\partial z}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right]}_{\text{shear production}}$$

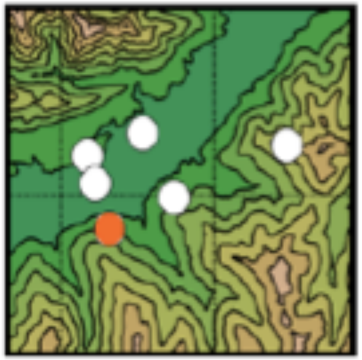
$$+ \underbrace{\frac{1}{\bar{\rho}}\frac{\partial}{\partial z}\left[\alpha_{\text{tke}}\bar{\rho}\lambda_l q \frac{\partial}{\partial z}\left(\frac{q^2}{2}\right)\right]}_{\text{vertical turbulent transport}} - \underbrace{\frac{q^3}{B_1 \lambda_l}}_{\text{dissipation}}$$

Example: TKE Budget Closure

Daytime TKE budget: **1D turbulence closure**



Steep slope



Afternoon:

- Vertical wind shear generation of TKE due to valley wind
- TKE underestimated

Example: TKE Budget Closure

- add horizontal terms (but not all): ‘hybrid TKE’
- advection
- horizontal shear production

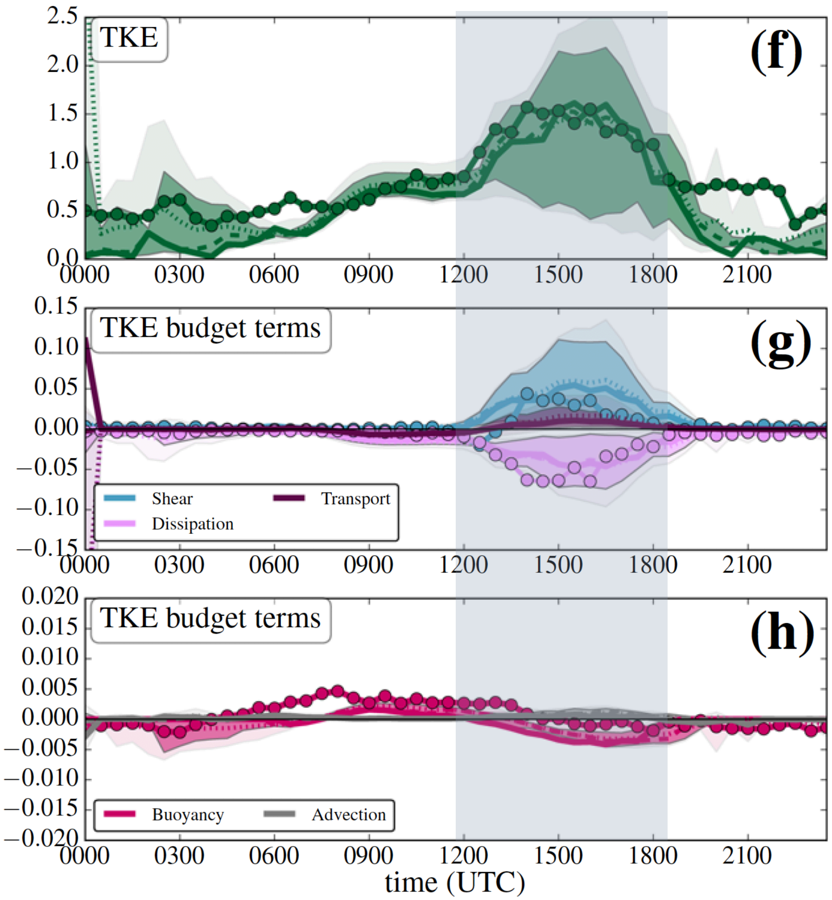
New Hybrid parametrization in the model:

$$\begin{aligned}
&\underbrace{\frac{\partial}{\partial t}\left(\frac{q^2}{2}\right)+\vec{U}\nabla\left(\frac{q^2}{2}\right)}_{\text{tendency \& advection}} = -\underbrace{K_H\frac{g}{\theta}\frac{\partial\theta}{\partial z}}_{\text{buoyancy production/consumption}} \\
&\underbrace{+(c\Delta x)^2\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2+\frac{1}{2}\left(\frac{\partial U}{\partial y}+\frac{\partial V}{\partial x}\right)^2\right]^{\frac{3}{2}}}_{\text{horizontal shear production}} \\
&+\underbrace{K_M\left[\left(\frac{\partial U}{\partial z}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2\right]}_{\text{vertical shear production}}+\underbrace{\frac{1}{\bar{\rho}}\frac{\partial}{\partial z}\left[\alpha_{\mathrm{tke}}\bar{\rho}\lambda_l q\frac{\partial}{\partial z}\left(\frac{q^2}{2}\right)\right]}_{\text{vertical turbulent transport}}-\underbrace{\frac{q^3}{B_1\lambda_l}}_{\text{dissipation}}
\end{aligned}$$

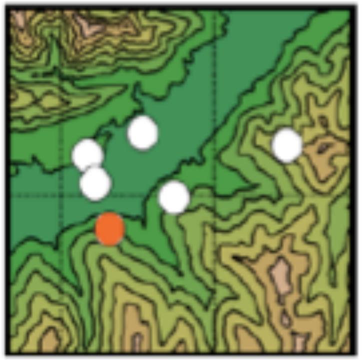
c... Smagorinsky constant, Δx... grid length

Example: TKE Budget Closure

Daytime TKE budget: **Hybrid turbulence closure**



Steep slope



Afternoon:

- 3D shear production
- Correct TKE simulation

TKE Budget

- measure for turbulence
- remember: 1.5 order closure (‘TKE closure’)
- in numerical models: description of turbulence (transport and mixing) is based on TKE (if not even simpler closure)
- most often: ‘BL approximation’
- basis for stability measures

Stability measures

TKE budget:

→ measure of production and damping of TKE

→ and therefore: of turbulence

→ if more TKE is produced than is removed

flow becomes more turbulent (more *dynamically* unstable)

→ if less TKE is produced than is removed

turbulence dies out

Stability Measures:

Static stability measure:

→ $\partial\overline{\theta}/\partial z$

Dynamic stability measure:

→ ratio of production/damping of TKE

Flux Richardson number

L.F. Richardson:

idealizing assumption:
- quasi-stationarity
- horizontally homogeneous
- no subsidence: $\overline{w}=0$
- coordinate system || mean wind

TKE-Budget:

$$0 = \underline{-\overline{u_1'u_3'}\frac{\partial \bar{u}_1}{\partial x_3}} - \frac{\partial \overline{u_3'e}}{\partial x_3} + \underline{\overline{u_3'\theta'}\frac{g}{\bar{\theta}}} - \frac{1}{\bar{\rho}}\frac{\partial \overline{u_3'p'}}{\partial x_3} - \varepsilon$$

assume: just these two are relevant

Flux Richardson number

Def: $$R_f = \frac{g}{\overline{\theta}} \frac{\overline{u'_3\theta'}}{\overline{u'_1u'_3}\frac{\partial \overline{u}_1}{\partial x_3}}$$

sign

<0

R_f = buoyancy term / shear production

neutral: $R_f = 0$

unstable: $R_f < 0$

stable: $R_f > 0$

$$\left.\begin{array}{l} \overline{u_3'\theta'} = 0 \\ \overline{u_3'\theta'} > 0 \\ \overline{u_3'\theta'} < 0 \end{array}\right\} \text{Static stability}$$

Flux Richardson number

$$R_f = \frac{g}{\overline{\theta}} \frac{\overline{u'_3 \theta'}}{\overline{u'_1 u'_3} \frac{\partial \overline{u}_1}{\partial x_3}}$$

Unstable (statically and dynamically):

→ friction *and* buoyancy contribute to production

→ no theoretical limit…

→ mostly > -10

Stable:

→ $R_f > 1$ damping > production (turbulence ceases)

→ $0 < R_f < 1$ statically stable,

dynamically unstable

(turbulence can exist)

Critical Richardson number

However: until now dissipation neglected!

production: shear production
damping: buoyancy + dissipation

‘critical’:

$$\frac{\frac{g}{\overline{\theta}}\overline{u'_3\,\theta'}-\varepsilon}{\overline{u'_1\,u'_3}\,\frac{\partial\overline{u}_1}{\partial x_3}}=1=R_f-\frac{\varepsilon}{\overline{u'_1\,u'_3}\,\frac{\partial\overline{u}_1}{\partial x_3}}\quad\rightarrow\quad R_{f,crit}=1+\frac{\varepsilon}{\overline{u'_1\,u'_3}\,\frac{\partial\overline{u}_1}{\partial x_3}}$$

Critical Richardson number

$$R_{f,crit} = 1 + \frac{\varepsilon}{\overline{u'_1 u'_3}\,\frac{\partial \overline{u}_1}{\partial x_3}}$$

$\rightarrow \varepsilon > 0$

$\rightarrow \overline{u'_1 u'_3}\,\frac{\partial \overline{u}_1}{\partial x_3} < 0$

→ 2nd term: always negative!

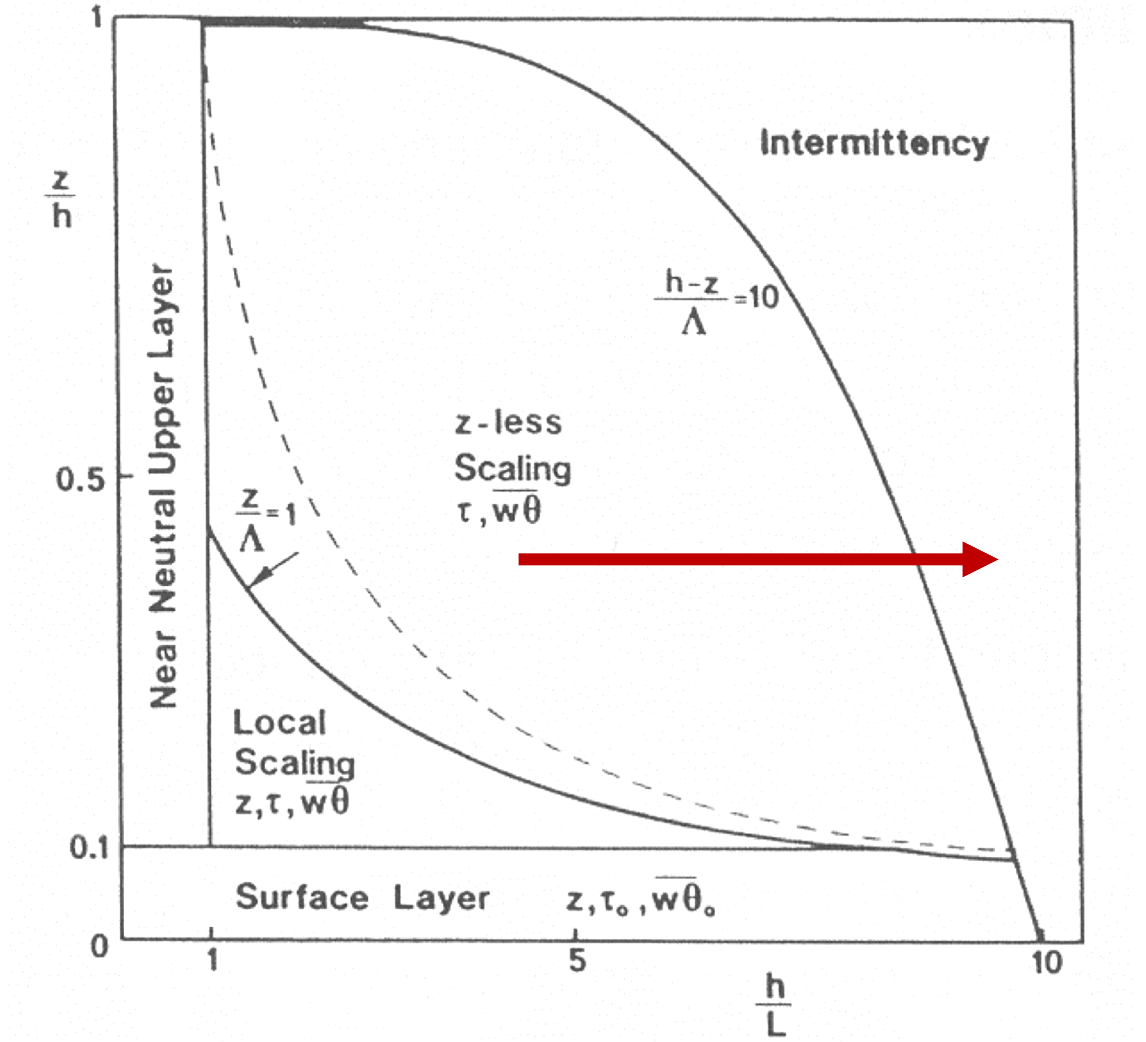
$$0 < R_{f,crit} < 1$$

→ larger Richardson number: intermittency

Critical Richardson number

Scaling regimes stable

→ larger R_f corresponds to intermittency



Critical Richardson number

$$0 < R_{f,crit} < 1$$

→ what is its value?

→ apparently dependent on the flow

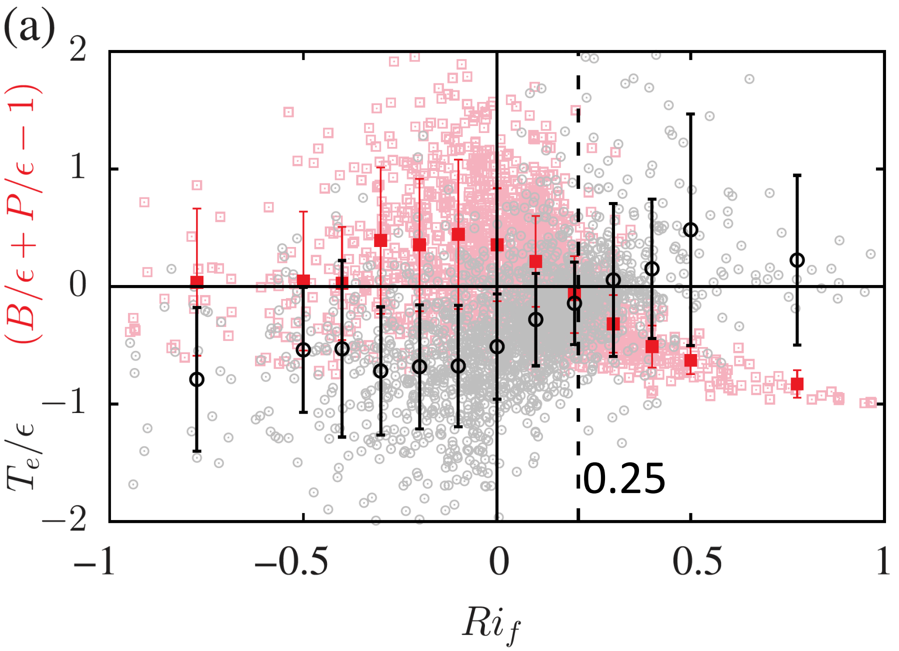
$$R_{f,crit} = 1 + \frac{\varepsilon}{\overline{u'_1 u'_3} \frac{\partial \overline{u}_1}{\partial x_3}}$$

→ often: $R_{f,crit} \approx 0.25$

Critical Richardson number

Limits of Applicability:

- We observe turbulence also for Rif > Ric
- Assumptions are not met: other terms of the equation important
- Need to modify $R_{f,crit}$ to include turbulent transport



Freire et al. 2019

Critical Richardson number

$$0 < R_{f,crit} < 1$$

→ what is its value?

→ apparently dependent on the flow

$$R_{f,crit} = 1 + \frac{\varepsilon}{\overline{u'_1 u'_3} \dfrac{\partial \overline{u}_1}{\partial x_3}}$$

→ often: $R_{f,crit} \approx 0.25$

→ even more often: $Ri_c \approx 0.25$

→ what is Ri (Ri_c)?

Gradient Richardson number

R_f: requires turbulent fluxes…

- → often not available
- → approximation with K-Theory
- → assumption: K_m=K_H

$$R_f = \frac{g}{\bar{\theta}} \frac{\overline{u_3'\theta'}}{\overline{u_1'u_3'}\,\frac{\partial \bar{u}_1}{\partial x_3}} \approx \frac{g}{\bar{\theta}} \frac{-\cancel{K}_H \frac{\partial \bar{\theta}}{\partial x_3}}{-\cancel{K}_M \frac{\partial \bar{u}_1}{\partial x_3} \cdot \frac{\partial \bar{u}_1}{\partial x_3}}$$

Def: $$Ri = \frac{g}{\bar{\theta}} \frac{(\partial \bar{\theta} / \partial x_3)}{(\partial \bar{u}_1 / \partial x_3)^2}$$

R_i: gradient Richardson number

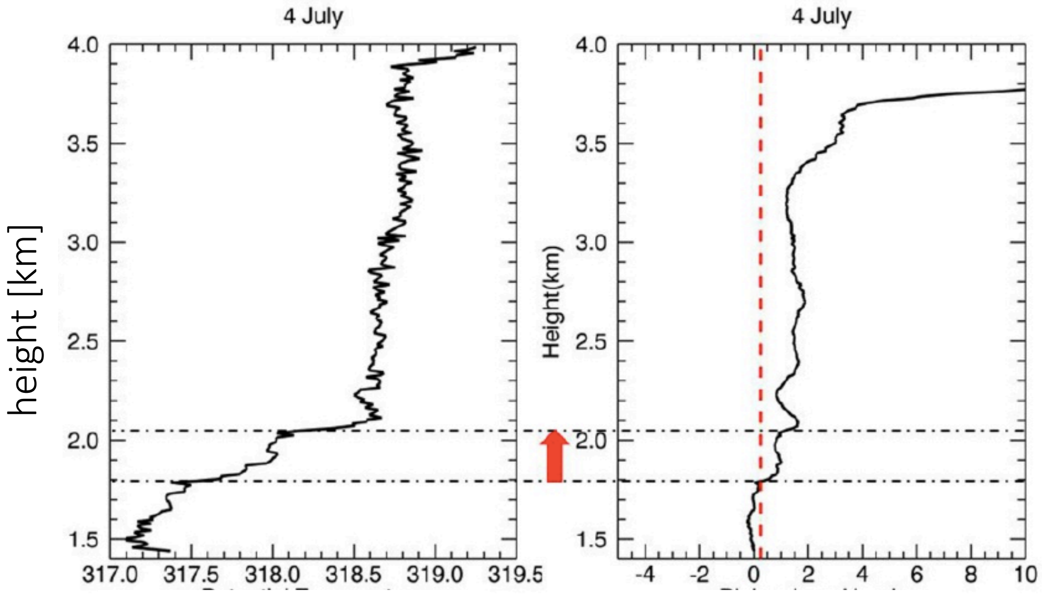
→ easier to determine than R_f

→ based on TKE-budget, too

Gradient Richardson number

- again: critical value, Ri_c (≈0.25)
  → for $Ri > Ri_c$: TKE damping > TKE production

- use Ri_c to find BL top (‘ABL is the layer of the atmosphere where turbulence dominates…’)



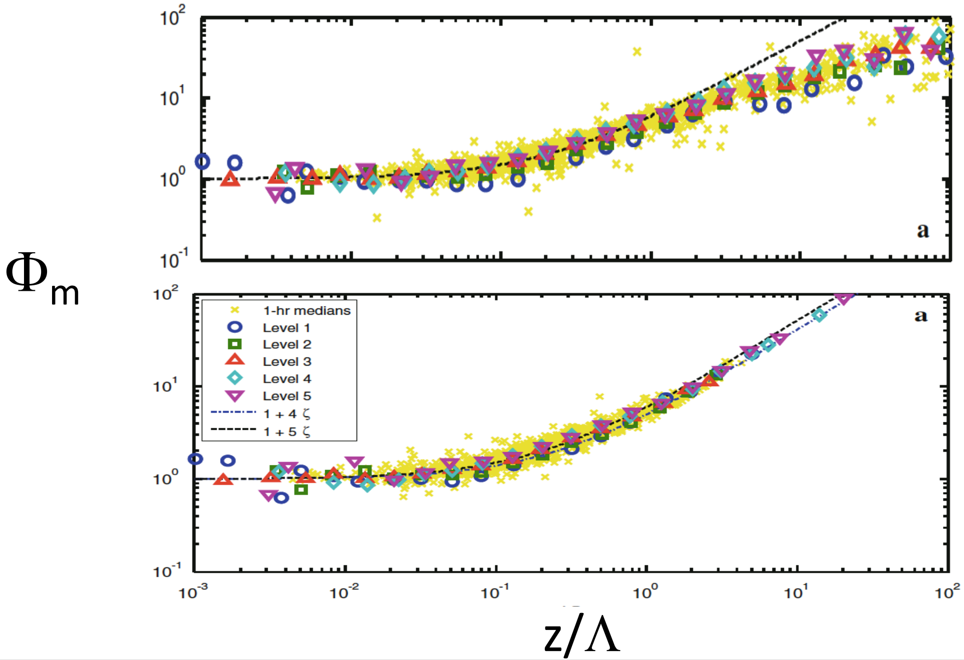
pot temperature [K]

Richardson number

Feng et al. (2015)

Gradient Richardson number

- Theoretically
  - → no critical value for gradient Richardson number
  - → often 0.25 or 0.21 are used
- Experiments: show turbulence at Ric > 0.21 behaves differently



Grachev et al. (2013)

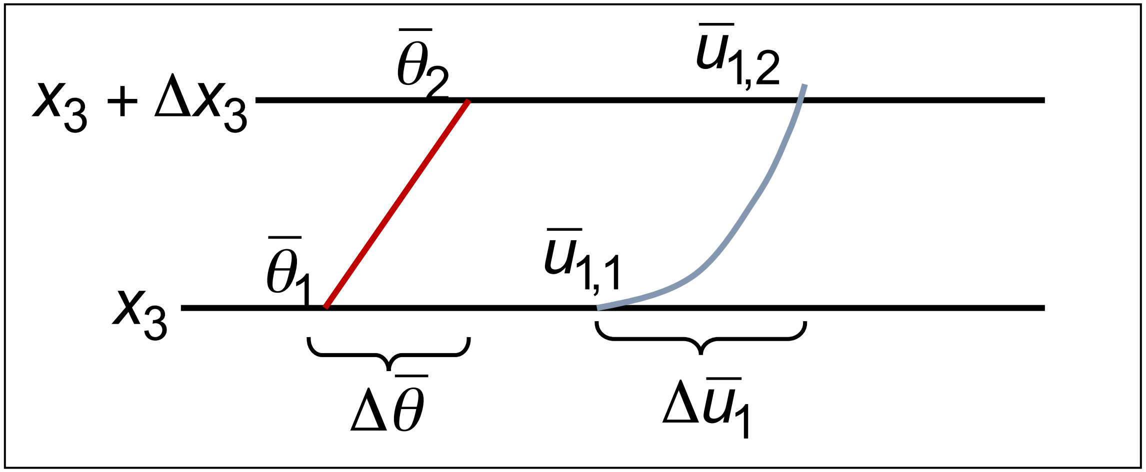
Bulk Richardson number

Even simpler:

→ gradients replaced by differences:

$$Ri = \frac{g}{\bar{\theta}} \frac{\partial\bar{\theta}/\partial x_3}{(\partial\bar{u}_1/\partial x_3)^2} \approx \frac{g}{\bar{\theta}} \frac{\Delta\bar{\theta}/\Delta x_3}{(\Delta\bar{u}_1/\Delta x_3)^2} = \boxed{\frac{g}{\bar{\theta}} \frac{\Delta\bar{\theta}\Delta x_3}{(\Delta\bar{u}_1)^2} = Ri_B}$$

→ *one value* for entire layer



→ often as measure of the stability for entire *PBL*

Stability in the surface layer

MOST: ‘everything ’ scales with z/L
→ TKE-budget as well

$$\boxed{\frac{\bar{a}}{a_*} = f_a(\frac{z}{L})}$$

TKE equation (non-dimensional):

- multiply with kx_3 / u_*^3
- replace turbulent fluxes by *surface fluxes*
- (quasi) stationary
- horizontally homogeneous

$$\frac{\partial \bar{e}}{\partial t} + \bar{u}_j \frac{\partial \bar{e}}{\partial x_j} = -\overline{u_i' u_j'} \frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \overline{u_j' e}}{\partial x_j} + \delta_{i3} \overline{u_i' \theta'} \frac{g}{\bar{\theta}} - \frac{1}{\bar{\rho}} \frac{\partial \overline{u_j' p'}}{\partial x_j} - \varepsilon$$

$$0 = -\overline{u_1' u_3'} \frac{\partial \bar{u}_1}{\partial x_3} - \frac{\partial \overline{u_3' e}}{\partial x_3} + \delta_{33} \overline{u_3' \theta'} \frac{g}{\bar{\theta}} - \frac{1}{\bar{\rho}} \frac{\partial \overline{u_3' p'}}{\partial x_3} - \varepsilon$$

Stability in the surface layer

$$0 = -\overline{u_1'u_3'}\frac{\partial \bar{u}_1}{\partial x_3}\frac{kx_3}{u_*^3} - \frac{\partial \overline{u_3'e}}{\partial x_3}\frac{kx_3}{u_*^3} + \overline{u_3'\theta'}\frac{g}{\bar{\theta}}\frac{kx_3}{u_*^3} - \frac{1}{\bar{\rho}}\frac{\partial \overline{u_3'p'}}{\partial x_3}\frac{kx_3}{u_*^3} - \varepsilon\frac{kx_3}{u_*^3}$$

$$0 = \frac{kx_3}{u_*}\frac{\partial \bar{u}_1}{\partial x_3} - \frac{kx_3}{u_*^3}\frac{\partial \overline{u_3'e}}{\partial x_3} - \frac{kx_3 g(\overline{u_3'\theta'})_o}{\bar{\theta}u_*^3} - \frac{kx_3}{\bar{\rho}u_*^3}\frac{\partial \overline{u_3'p'}}{\partial x_3} - \frac{kx_3}{u_*^3}\varepsilon$$

$$= \quad \Phi_m \quad - \quad \Phi_{tr} \quad + \quad \frac{z}{L} \quad - \quad \Phi_p \quad - \quad \Phi_\varepsilon$$

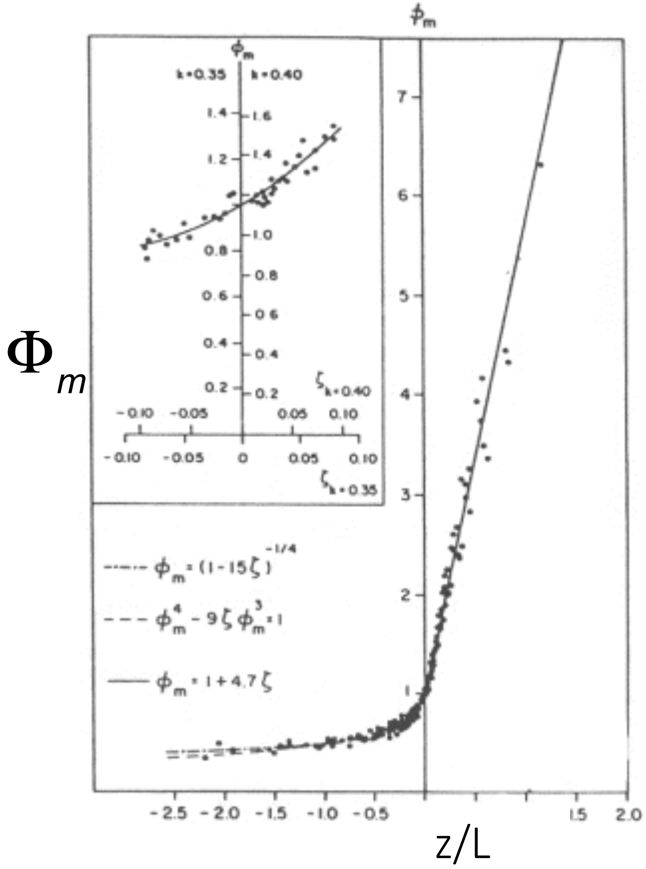
→ each term a function of z/L

→ z/L: stability measure

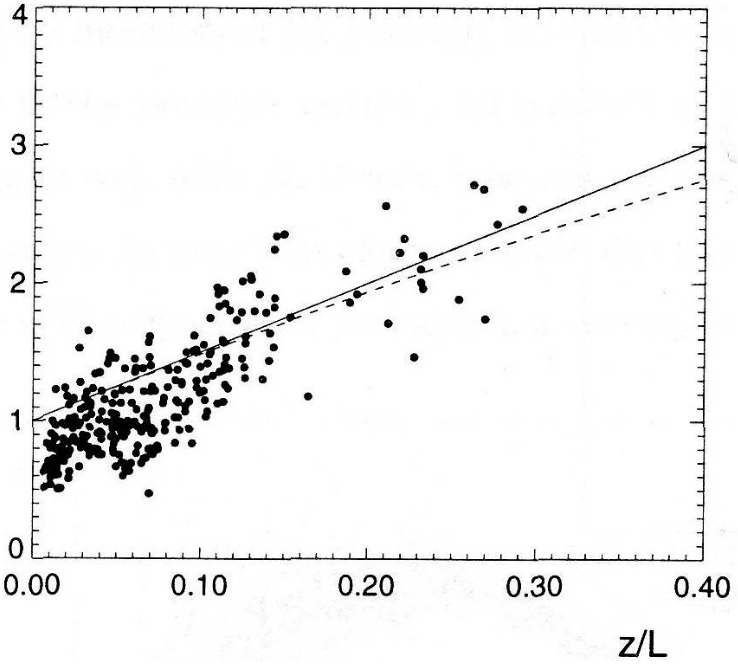
Stability in the surface layer

z/L: also corresponds to ratio buoyancy / shear production

all Φ functions: dependent on z/L (only one π-group)

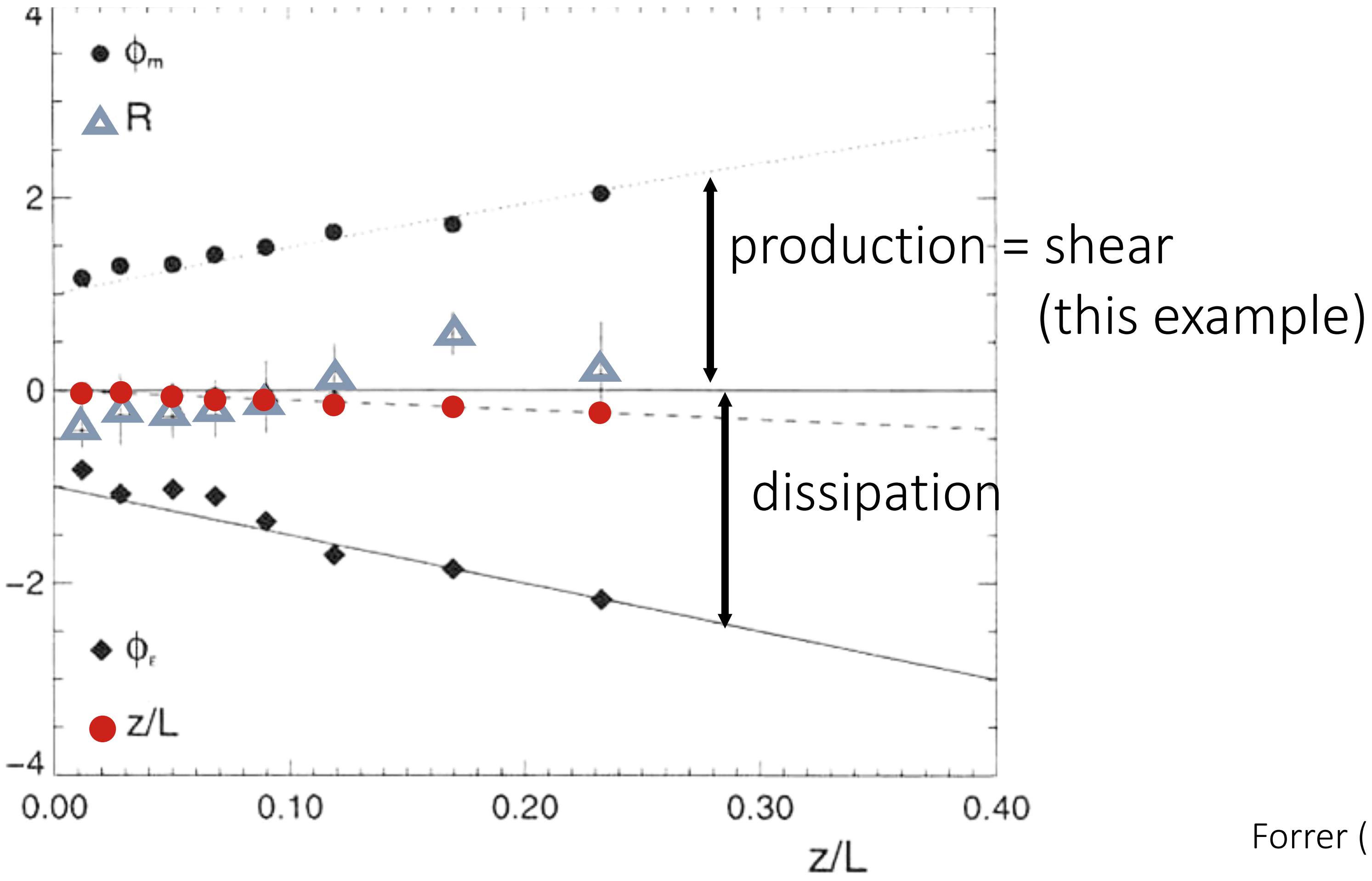


$\Phi\varepsilon$: Scaled Dissipation



TKE budget in the Surface layer

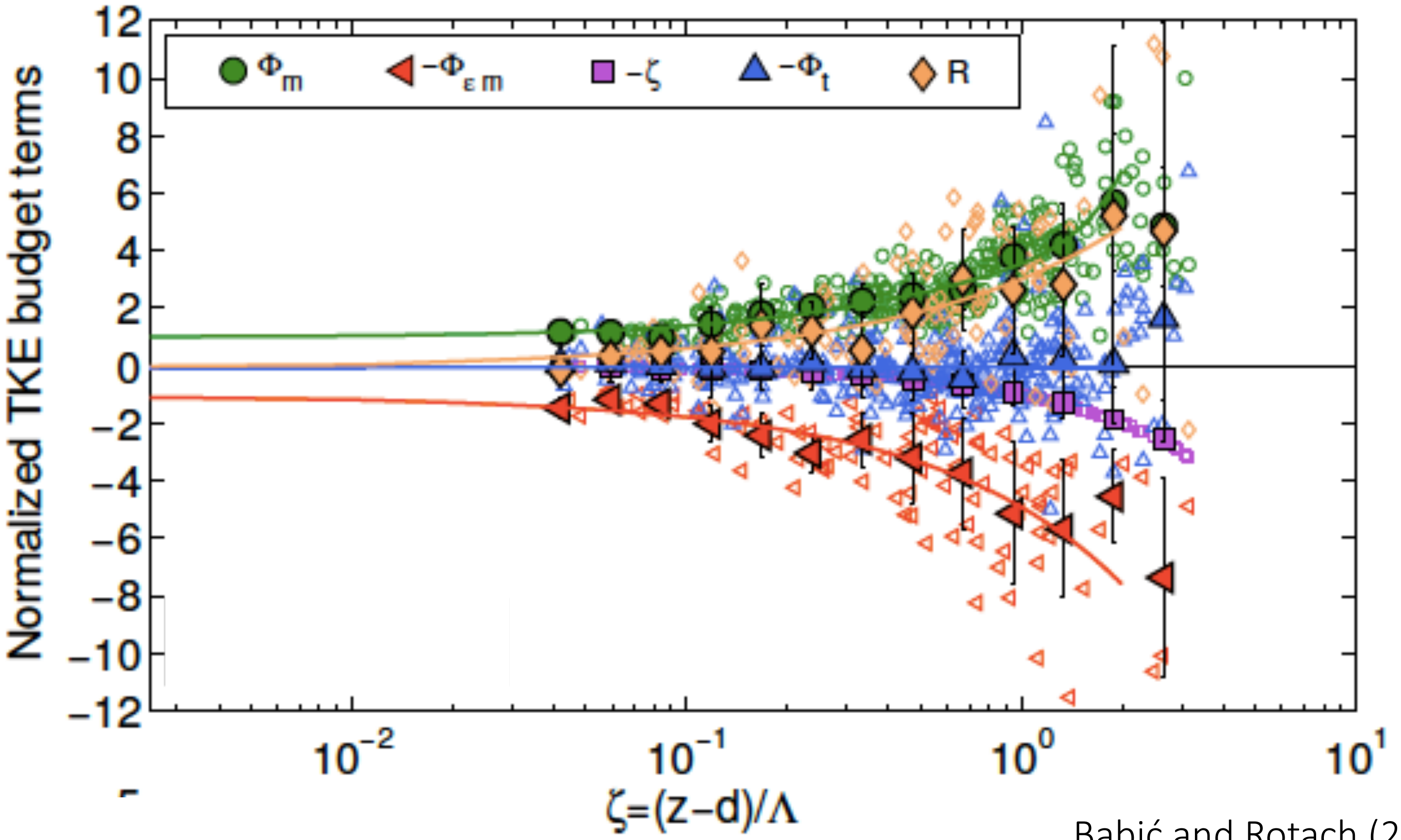
Scaled Components of TKE budget in SL as function of z/L



Forrer (1997)

TKE budget in the Surface layer

→ transition layer over a forest stand



Babić and Rotach (2018)

Turbulent potential energy

Classically:
→ In stable conditions when Rf > Rfc
→ local shear cannot maintain turbulence
→ flow becomes laminar

Zilitinkevich et al. 2008:
→ Turbulence can exist beyond critical R_f
→ We need to examine also Turbulent Potential energy
→ It is Total Turbulent energy (TTE) that determines if mixing is strong or weak (not turbulent and laminar)
→ In atmosphere and ocean turbulence persists to Ri >> 1

Turbulent potential energy

Turbulent potential energy

$$E_p = \left(\frac{g}{\theta N}\right)^2 \frac{1}{2}\overline{\theta'^2}$$

Budget equation for TPE

$$\frac{DE_p}{Dt} + \boxed{\frac{\partial}{\partial z}\left(\left(\frac{g}{\theta N}\right)^2 \overline{\theta'^2 w'}\right)} = \boxed{-\left(\frac{g}{\theta N}\right)^2 \overline{\theta' w'}} - \boxed{\left(\frac{g}{\theta N}\right)^2 \epsilon_\theta}$$

Dissipation

Transport

Potential temperature flux

Link to TKE equation

Summary

TKE conservation equation:
→ shear production
→ buoyancy
→ transport (TKE and pressure)
→ dissipation
→ all this: in the BL approximation (vertical)

TKE budget as basis for
→ *dynamical* stability measures
→ R_f, Ri, R_B
→ Surface layer: z/L

More accurate way of modeling stable turbulence
→ TKE + TPE approach