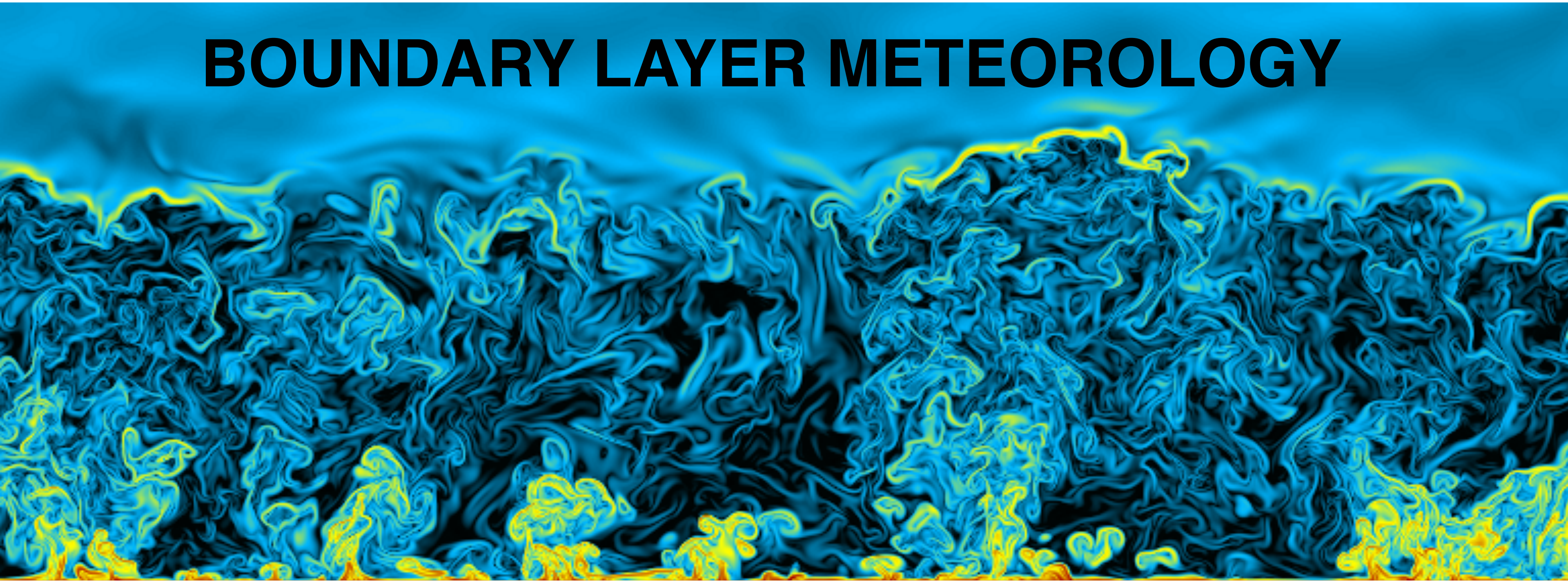
universität innsbruck

BOUNDARY LAYER METEOROLOGY



Prof. Ivana Stiperski, Dr. Manuela Lehner

Department of Atmospheric and Cryospheric Sciences

Chapter 5

Conservation Equations of Turbulent flows

Content

Chapter 5.1

- Conservation equations for Mean Variables

- Conservation equations for Turbulent Variables

Chapter 5.2

- Closure problem and Closure schemes

Statistical Treatment of Turbulence

Statistical Treatment of Turbulence:

→Reynolds decomposition and - averaging

→ conservation equations:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\delta_{i3} g + f_c \varepsilon_{ij3} u_j - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j}$$

$$\frac{\partial \overline{u_i}}{\partial t} + \overline{u_j} \frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\overline{u_i' u_j'}\right) = -\delta_{i3} g + f_c \varepsilon_{ij3} \overline{u_j} - \frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial^2 \overline{u_i}}{\partial x_j^2}$$

→ new variables....

2nd Order Moments

→ 2 approaches for treating these new variables

I: Physical approach:
further development of conservation equations

→ simplify (assumptions), solve
→ numerical solutions
→ higher order

II: Similarity theory

$$\overline{w'\theta'},\ \overline{w'q'} \text{ and } \overline{u'w'}, \ldots$$

→ scale analysis
→ *characteristics* of the result?

→ I + II combined (e.g. numerical models, often)

Starting point

Turbulent scales:
→ millimeter
→ seconds

chapter on spectral characteristics

numerical modeling:
→ resolution **milli**meters.....
→ indeed being done (DNS)
→ extremely CPU time consuming

→ **statistical treatment:**
Reynolds decomposition and – averaging
to separate turbulence and mean

Procedure

Conservation Equations for mean variables

Step 1:	Formulate the conservation equation (Table 5.1).
Step 2:	Simplify where appropriate.
Step 3:	Apply Reynolds decomposition.
Step 4:	Reynolds-average the resulting equation to obtain the *conservation equation for the mean flow variable*.
Step 5:	Express the result of step 4 in *flux form* (if appropriate).
Step 6:	Subtract the result of step 4 from that of step 3 to obtain the *conservation equation for the fluctuating variable*.
Step 7:	Multiply the result of step 6 with another fluctuating variable to obtain an equation for the corresponding second order moment.
Step 8:	Reynolds-average the result of step 7 to obtain the *conservation equation for this mean second order moment*.
(Step 9:	Repeat steps 6-8 to obtain conservation equations for even higher order moments).

Conservation Equations for higher order moments

Conservation equations

Momentum:	$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\delta_{i3} g + f_c \varepsilon_{ij3} u_j - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j}$	(5.1a)
Energy:	$\frac{\partial \theta}{\partial t} + u_j \frac{\partial \theta}{\partial x_j} = \nu_\theta \frac{\partial^2 \theta}{\partial x_j^2} - \frac{1}{\rho c_p}\frac{\partial NR_j}{\partial x_j} - \frac{L_v E}{\rho c_p} + \frac{R_c}{\rho c_p}$	(5.1b)
Specific humidity:	$\frac{\partial q}{\partial t} + u_j \frac{\partial q}{\partial x_j} = \frac{E}{\rho}$	(5.1c)
Trace gas:	$\frac{\partial C}{\partial t} + u_j \frac{\partial C}{\partial x_j} = Q - S$	(5.1d)
Mass:	$\frac{\partial \rho}{\partial t} + u_j \frac{\partial \rho}{\partial x_j} = -\rho \frac{\partial u_j}{\partial x_j}$	(5.1e)
Equation of State:	$p = R_a \cdot \rho \cdot T$	(5.1f)

Equation of state

$$p = R_a \cdot \rho \cdot T \qquad\qquad R_a = 286 Jkg^{-1}K^{-1}$$

→ Reynolds decomposition

$$\left.\begin{aligned} p &= \bar{p} + p' \\ \rho &= \bar{\rho} + \rho' \\ T &= \bar{T} + T' \end{aligned}\right\} \quad \text{insert}$$

$$\overline{\bar{p} + \cancelto{0}{p'}} = R_a(\overline{\bar{\rho}\bar{T}} + \cancelto{0}{\overline{\bar{\rho}T'}} + \cancelto{0}{\overline{\rho'\bar{T}}} + \overline{\rho' T'})$$

→ average again:

$$\bar{p} = R_a(\bar{\rho}\bar{T} + \overline{\rho' T'})$$

Equation of state

$$\overline{p} = R_a(\overline{\rho}\,\overline{T} + \overline{\rho' T'}) \longleftarrow \overline{\rho}\,\overline{T} \gg \overline{\rho' T'}$$

$$\boxed{\overline{p} \approx R_a(\overline{\rho}\,\overline{T})}$$

To get fluctuations p':

$$\cancel{\overline{p}} + p' = R_a(\cancel{\overline{\rho}\,\overline{T}} + \overline{\rho}T' + \rho'\overline{T} + \rho' T')$$

$$p' = R_a(\overline{\rho}T' + \rho'\overline{T} + \rho' T')$$

$$p'/R_a = (\overline{\rho}T' + \rho'\overline{T} + \rho' T')$$

divide by

$$\overline{p}/R_a = \overline{\rho}\,\overline{T}$$

$$\boxed{\frac{p'}{\overline{p}} = \frac{T'}{\overline{T}} + \frac{\rho'}{\overline{\rho}} + \frac{\rho' T'}{\overline{\rho}\,\overline{T}}}$$

Equation of state

$$\boxed{\frac{p'}{\overline{p}} = \frac{T'}{\overline{\overline{T}}} + \frac{\rho'}{\overline{\rho}} + \cancelto{0}{\frac{\rho' T'}{\overline{\rho}\,\overline{\overline{T}}}}} \longleftarrow \frac{T'}{\overline{\overline{T}}} << 1, \quad \frac{\rho'}{\overline{\rho}} << 1$$

$$p' = \mathcal{O}\,(0.1\,\text{hPa}) \Rightarrow p'/\overline{p} = \mathcal{O}(10^{-4})$$

$$T' = \mathcal{O}\,(1\text{K}) \Rightarrow T'/\overline{T} = \mathcal{O}(10^{-2})$$

therefore must:

$$\rho'/\overline{\rho} \;=\; \mathcal{O}(10^{-2})$$

$$\longrightarrow \boxed{\frac{\rho'}{\overline{\rho}} \approx -\frac{T'}{\overline{\overline{T}}}}$$

→ replace $\frac{\rho'}{\overline{\rho}}$ by $\frac{T'}{\overline{\overline{T}}}$

→ easier to determine

Conservation equations

Momentum:	$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\delta_{i3} g + f_c \varepsilon_{ij3} u_j - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j}$	(5.1a)
Energy:	$\frac{\partial \theta}{\partial t} + u_j \frac{\partial \theta}{\partial x_j} = \nu_\theta \frac{\partial^2 \theta}{\partial x_j^2} - \frac{1}{\rho c_p}\frac{\partial NR_j}{\partial x_j} - \frac{L_v E}{\rho c_p} + \frac{R_c}{\rho c_p}$	(5.1b)
Specific humidity:	$\frac{\partial q}{\partial t} + u_j \frac{\partial q}{\partial x_j} = \frac{E}{\rho}$	(5.1c)
Trace gas:	$\frac{\partial C}{\partial t} + u_j \frac{\partial C}{\partial x_j} = Q - S$	(5.1d)
Mass:	$\frac{\partial \rho}{\partial t} + u_j \frac{\partial \rho}{\partial x_j} = -\rho \frac{\partial u_j}{\partial x_j}$	(5.1e)
Equation of State:	$p = R_a \cdot \rho \cdot T$	(5.1f)

Conservation of mass

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_j)}{\partial x_j} = 0 \longleftrightarrow \frac{d\rho}{dt} + \rho \frac{\partial u_j}{\partial x_j} = 0$$

incompressibility:

$$\frac{{d\rho}/{dt}}{\rho} << \frac{\partial u_1}{\partial x_1}, \frac{\partial u_2}{\partial x_2}, \frac{\partial u_3}{\partial x_3}$$

$$\longrightarrow \boxed{\frac{\partial u_j}{\partial x_j} \approx 0} \longrightarrow$$

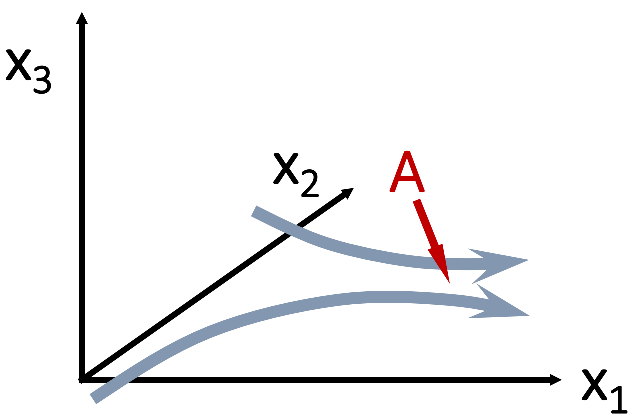
Reynolds-decomposition and averaging

$$\boxed{\frac{\partial \overline{u}_j}{\partial x_j} = 0 \quad ; \frac{\partial u'_j}{\partial x_j} = 0}$$

Conservation of mass: Incompressibility

→ let: $\dfrac{\partial u_i(\vec{x},t)}{\partial x_i} = 0$ summed!

→ special form of mass conservations equation
→ Atmosphere: ‘always and everywhere’ fulfilled



possibilities:

1. air exits towards the top
2. flow accelerates
3. density in A increases

1 & 2: incompressible flow

universität innsbruck

Incompressibility: 'Flux Form'

Conservation equations

→ Advection term: $u_j \frac{\partial \chi}{\partial x_j}$

identity: $$\frac{\partial (u_j \chi)}{\partial x_j} = \chi \underbrace{\cancel{\frac{\partial u_j}{\partial x_j}}}_{0} + u_j \frac{\partial \chi}{\partial x_j}$$

$$\rightarrow u_j \frac{\partial \chi}{\partial x_j} = \frac{\partial (u_j \chi)}{\partial x_j}$$

advection term corresponds to flux divergence (average and fluctuations)

in particular:

$$\overline{u'_j \frac{\partial \chi'}{\partial x_j}} = \frac{\partial \left(\overline{u'_j \chi'}\right)}{\partial x_j}$$

'turbulent advection' term:
→ divergence of turbulent flux

Conservation equations

Momentum:	$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\delta_{i3} g + f_c \varepsilon_{ij3} u_j - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j}$	(5.1a)
Energy:	$\frac{\partial \theta}{\partial t} + u_j \frac{\partial \theta}{\partial x_j} = \nu_\theta \frac{\partial^2 \theta}{\partial x_j^2} - \frac{1}{\rho c_p}\frac{\partial NR_j}{\partial x_j} - \frac{L_v E}{\rho c_p} + \frac{R_c}{\rho c_p}$	(5.1b)
Specific humidity:	$\frac{\partial q}{\partial t} + u_j \frac{\partial q}{\partial x_j} = \frac{E}{\rho}$	(5.1c)
Trace gas:	$\frac{\partial C}{\partial t} + u_j \frac{\partial C}{\partial x_j} = Q - S$	(5.1d)
Mass:	$\frac{\partial \rho}{\partial t} + u_j \frac{\partial \rho}{\partial x_j} = -\rho \frac{\partial u_j}{\partial x_j}$	(5.1e)
Equation of State:	$p = R_a \cdot \rho \cdot T$	(5.1f)

Conservation of momentum

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\delta_{i3} g + f_c \varepsilon_{ij3} u_j - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j}$$

I II III IV V VI

→ equations: u, v, w

I: local temporal change

II: advection

I + II = total (Lagrangian) temporal change

III: gravity $\neq 0$ for $i = 3$

IV: Coriolis term

V: pressure gradient acceleration

VI: molecular friction

Procedure

Step 1:	Formulate the conservation equation (Table 5.1).
Step 2:	Simplify where appropriate.
Step 3:	Apply Reynolds decomposition.
Step 4:	Reynolds-average the resulting equation to obtain the *conservation equation for the mean flow variable*.
Step 5:	Express the result of step 4 in *flux form* (if appropriate).
Step 6:	Subtract the result of step 4 from that of step 3 to obtain the *conservation equation for the fluctuating variable*.
Step 7:	Multiply the result of step 6 with another fluctuating variable to obtain an equation for the corresponding second order moment.
Step 8:	Reynolds-average the result of step 7 to obtain the *conservation equation for this mean second order moment*.
(Step 9:	Repeat steps 6-8 to obtain conservation equations for even higher order moments).

Simplifications

1. Newtonian fluid: Term VI → $$\frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j} = \nu \frac{\partial^2 u_i}{\partial x_j^2}$$

2. Boussinesq Approximation

$$\frac{\rho'}{\overline{\rho}} << 1$$ → density fluctuations negligible

but: one term $\frac{\rho'}{\overline{\rho}} g$ is not negligible

Bussinesq approximation

In practice this means:

1. replace $\rho \rightarrow \overline{\rho}$

2. replace $g \rightarrow g(1-\theta'/\overline{\theta})$

It is valid if

✓ • PBL height small with respect to 'scale Height' (approx. 8km)

✓ $\frac{T'}{\overline{\overline{T}}} << 1, \quad \frac{\rho'}{\overline{\rho}} << 1$

✓ • Flow is incompressible

(✓) • Stratification is not too extremely stable

Conservation of momentum

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\delta_{i3} g + f_c \varepsilon_{ij3} u_j - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j}$$

simplifications:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\delta_{i3} g \left(1 - \frac{\theta'}{\bar{\theta}}\right) + f_c \varepsilon_{ij3} u_j - \frac{1}{\bar{\rho}}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2}$$

Reynolds decomposition and averaging:

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} + \overline{u'_j \frac{\partial u'_i}{\partial x_j}} = -\delta_{i3} g + f_c \varepsilon_{ij3} \bar{u}_j - \frac{1}{\bar{\rho}}\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j^2}$$

→ 'overbars' (averaging)

→ one new term (three, actually – summed)

→ density fluctuations disappeared….

Conservation of momentum

Flux form:

$$\boxed{\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} + \boxed{\frac{\partial}{\partial x_j}\left(\overline{u_i' u_j'}\right)} = -\delta_{i3} g + f_c \varepsilon_{ij3} \bar{u}_j - \frac{1}{\bar{\rho}} \frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j^2}}$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\delta_{i3} g + f_c \varepsilon_{ij3} u_j - \frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{1}{\rho} \frac{\partial \sigma_{ij}}{\partial x_j}$$

Original equation:

Reynolds decomposition: (i.e. introduction of turbulence)

→ new term(s): flux divergence

Conservation equations

Momentum:	$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\delta_{i3} g + f_c \varepsilon_{ij3} u_j - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j}$$	(5.1a)
Energy:	$$\frac{\partial \theta}{\partial t} + u_j \frac{\partial \theta}{\partial x_j} = \nu_\theta \frac{\partial^2 \theta}{\partial x_j^2} - \frac{1}{\rho c_p}\frac{\partial NR_j}{\partial x_j} - \frac{L_v E}{\rho c_p} + \frac{R_c}{\rho c_p}$$	(5.1b)
Specific humidity:	$$\frac{\partial q}{\partial t} + u_j \frac{\partial q}{\partial x_j} = \frac{E}{\rho}$$	(5.1c)
Trace gas:	$$\frac{\partial C}{\partial t} + u_j \frac{\partial C}{\partial x_j} = Q - S$$	(5.1d)
Mass:	$$\frac{\partial \rho}{\partial t} + u_j \frac{\partial \rho}{\partial x_j} = -\rho \frac{\partial u_j}{\partial x_j}$$	(5.1e)
Equation of State:	$$p = R_a \cdot \rho \cdot T$$	(5.1f)

Conservation of energy

→ Energy conservation corresponds to conservation of θ

$$\frac{\partial\theta}{\partial t}+u_j\frac{\partial\theta}{\partial x_j}=\nu_\theta\frac{\partial^2\theta}{\partial x_j^2}-\frac{1}{\rho c_p}\frac{\partial NR_j}{\partial x_j}-\frac{L_v E}{\rho c_p}+\frac{R_c}{\rho c_p}$$

I II III IV V VI

I & II: total temporal change

III: molecular diffusion of heat

IV: radiation divergence

V: phase change of water

VI: 'everything else' (chemical reactions, antropogenic sources..)

Conservation of energy

$$\frac{\partial \theta}{\partial t} + u_j \frac{\partial \theta}{\partial x_j} = \nu_\theta \frac{\partial^2 \theta}{\partial x_j^2} \underbrace{- \frac{1}{\rho c_p} \frac{\partial NR_j}{\partial x_j} - \frac{L_v E}{\rho c_p} + \frac{R_c}{\rho c_p}}_{\equiv \frac{1}{\rho c_p} QR}$$

→ all 'external' processes

Reynolds decomposition and averaging:

$$\boxed{\frac{\partial \overline{\theta}}{\partial t} + \overline{u}_j \frac{\partial \overline{\theta}}{\partial x_j} + \boxed{\frac{\partial}{\partial x_j} \left(\overline{u_j' \theta'}\right)} = \nu_\theta \frac{\partial^2 \overline{\theta}}{\partial x_j^2} + \frac{1}{\overline{\rho} c_p} \overline{QR}}$$

Conservation of scalars

Water vapor:

$$\frac{\partial \overline{q}}{\partial t} + \overline{u}_j \frac{\partial \overline{q}}{\partial x_j} + \boxed{\frac{\partial}{\partial x_j}(\overline{u_j' q'})} = \frac{\overline{E}}{\overline{\rho}}$$

'Tracer constituents':

$$\frac{\partial \overline{C}}{\partial t} + \overline{u}_j \frac{\partial \overline{C}}{\partial x_j} + \boxed{\frac{\partial}{\partial x_j}(\overline{u_j' C'})} = \overline{Q} - \overline{S}$$

Conservation equations for mean variables in turbulent flow

Momentum:	$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} + \boxed{\frac{\partial}{\partial x_j}\left(\overline{u_i' u_j'}\right)} = -\delta_{i3} g + f_c \varepsilon_{ij3} \overline{u}_j - \frac{1}{\overline{\rho}} \frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial^2 \overline{u}_i}{\partial x_j^2}$	(5.16)
Energy:	$\frac{\partial \overline{\theta}}{\partial t} + \overline{u}_j \frac{\partial \overline{\theta}}{\partial x_j} + \boxed{\frac{\partial}{\partial x_j}\left(\overline{u'_j \theta'}\right)} = \nu_\theta \frac{\partial^2 \overline{\theta}}{\partial x_j^2} + \frac{1}{\overline{\rho} c_p} \overline{QR}$	(5.23)
Specific humidity:	$\frac{\partial \overline{q}}{\partial t} + \overline{u}_j \frac{\partial \overline{q}}{\partial x_j} + \boxed{\frac{\partial}{\partial x_j}\left(\overline{u_j' q'}\right)} = \frac{\overline{E}}{\overline{\rho}}$	(5.24)
Trace gas:	$\frac{\partial \overline{C}}{\partial t} + \overline{u}_j \frac{\partial \overline{C}}{\partial x_j} + \boxed{\frac{\partial}{\partial x_j}\left(\overline{u_j' C'}\right)} = \overline{Q} - \overline{S}$	(5.25)
Mass:	$\frac{\partial \overline{u}_j}{\partial x_j} = 0 \quad ; \quad \frac{\partial u'_j}{\partial x_j} = 0$	(5.8)
Equation of State:	$\overline{p} = R_a \overline{\rho} \overline{T}$	(5.4)

Summary

- Reynolds decomposition and averaging
- new term: divergence of the **turbulent fluxes**
- from advection terms: $u_1 \frac{\partial u_i}{\partial x_1},\ u_2 \frac{\partial u_i}{\partial x_2},\ u_3 \frac{\partial u_i}{\partial x_3}$

$$\underbrace{\overline{u}_1 \frac{\partial \overline{u_i}}{\partial x_1},\ \overline{u}_2 \frac{\partial \overline{u_i}}{\partial x_2}}_{\text{horiz. homogenous} \rightarrow =0},\ \underbrace{\overline{u}_3 \frac{\partial \overline{u_i}}{\partial x_3}}_{\overline{u}_3=0 \rightarrow =0} \qquad \underbrace{\frac{\partial}{\partial x_1}(\overline{u_i' u_1'}),\ \frac{\partial}{\partial x_2}(\overline{u_i' u_2'})}_{\text{horiz. homogenous} \rightarrow =0},\ \frac{\partial}{\partial x_3}(\overline{u_i' u_3'})$$

$\longrightarrow \overline{u'w'},\ \overline{v'w'}$ are important

$\longrightarrow \overline{w'q'},\ \overline{w'\theta'}$ the same

their vertical divergence!

Procedure

Conservation Equations for mean variables

→ In each of the conservation equations:

→ new terms: flux divergences

→ find conservation equations for these new variables

→ example: second moments

Step 6:	Subtract the result of step 4 from that of step 3 to obtain the *conservation equation for the fluctuating variable*.
Step 7:	Multiply the result of step 6 with another fluctuating variable to obtain an equation for the corresponding second order moment.
Step 8:	Reynolds-average the result of step 7 to obtain the *conservation equation for this mean second order moment*.
(Step 9:	Repeat steps 6-8 to obtain conservation equations for even higher order moments).

Conservation Equations for higher order moments

Higher order moments: e.g. momentum

completely expanded equation:

$$\frac{\partial \overline{u}_i}{\partial t} + \frac{\partial u_i'}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} + \overline{u}_j \frac{\partial u_i'}{\partial x_j} + u_j' \frac{\partial \overline{u}_i}{\partial x_j} + u_j' \frac{\partial u_i'}{\partial x_j} =$$

$$-\delta_{i3} g + \delta_{i3} \left(\frac{\theta'}{\overline{\theta}} \right) g + f_c \varepsilon_{ij3} \overline{u}_j + f_c \varepsilon_{ij3} u_j' - \frac{1}{\overline{\rho}} \frac{\partial \overline{p}}{\partial x_i} - \frac{1}{\overline{\rho}} \frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 \overline{u}_i}{\partial x_j^2} + \nu \frac{\partial^2 u_i'}{\partial x_j^2}$$

subtract averaged equation:

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial}{\partial x_j} (\overline{u_i' u_j'}) = -\delta_{i3} g + f_c \varepsilon_{ij3} \overline{u}_j - \frac{1}{\overline{\rho}} \frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial^2 \overline{u}_i}{\partial x_j^2}$$

→ result: ./.

Higher order moments: e.g. momentum

→ conservation equations for fluctuation:

$$\frac{\partial u_i'}{\partial t} + \overline{u}_j \frac{\partial u_i'}{\partial x_j} + u_j' \frac{\partial \overline{u}_i}{\partial x_j} + u_j' \frac{\partial u_i'}{\partial x_j} - \frac{\partial \overline{u_i' u_j'}}{\partial x_j} = + \delta_{i3}\left(\frac{\theta'}{\overline{\theta}}\right) g + f_c \varepsilon_{ij3} u_j' - \frac{1}{\overline{\rho}} \frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 u_i'}{\partial x_j^2}$$

multiply by $2u_i'$

$$2u_i' \frac{\partial u_i'}{\partial t} + 2\overline{u}_j u_i' \frac{\partial u_i'}{\partial x_j} + 2u_i' u_j' \frac{\partial \overline{u}_i}{\partial x_j} + 2u_i' u_j' \frac{\partial u_i'}{\partial x_j} - 2u_i' \frac{\partial \overline{u_i' u_j'}}{\partial x_j} = + 2\delta_{i3} u_i' \left(\frac{\theta'}{\overline{\theta}}\right) g + 2 f_c \varepsilon_{ij3} u_i' u_j' - 2 \frac{u_i'}{\overline{\rho}} \frac{\partial p'}{\partial x_i} + 2\nu u_i' \frac{\partial^2 u_i'}{\partial x_j^2}$$

! summation introduced

Higher order moments: e.g. momentum

→ 'cosmetics'

$$\frac{\partial u_i'^2}{\partial t} + \bar{u}_j \frac{\partial u_i'^2}{\partial x_j} + 2u_i'u_j' \frac{\partial \bar{u}_i}{\partial x_j} + u_j' \frac{\partial u_i'^2}{\partial x_j} - 2u_i' \frac{\partial \overline{u_i'u_j'}}{\partial x_j} =$$

$$+2\delta_{i3} u_i' \left(\frac{\theta'}{\bar{\theta}} \right) g + 2f_c \varepsilon_{ij3} u_i'u_j' - 2\frac{u_i'}{\bar{\rho}} \frac{\partial p'}{\partial x_i} + 2\nu u_i' \frac{\partial^2 u_i'}{\partial x_j^2}$$

still summed!

→ Reynolds averaging

$$\frac{\partial \overline{u_i'^2}}{\partial t} + \bar{u}_j \frac{\partial \overline{u_i'^2}}{\partial x_j} + 2\overline{u_i'u_j'} \frac{\partial \bar{u}_i}{\partial x_j} + \overline{u_j' \frac{\partial u_i'^2}{\partial x_j}} =$$

$$+2\delta_{i3} \overline{u_i' \left(\frac{\theta'}{\bar{\theta}} \right)} g + 2f_c \varepsilon_{ij3} \overline{u_i'u_j'} - 2\overline{\frac{u_i'}{\bar{\rho}} \frac{\partial p'}{\partial x_i}} + 2\nu \overline{u_i' \frac{\partial^2 u_i'}{\partial x_j^2}}$$

still summed!

Higher order moments: e.g. momentum

→ flux form

$$\boxed{\begin{aligned}&\frac{\partial \overline{u_i'^2}}{\partial t}+\bar{u}_j\frac{\partial \overline{u_i'^2}}{\partial x_j}=-2\overline{u_i'u_j'}\frac{\partial \bar{u}_i}{\partial x_j}-\frac{\partial \overline{u_j'u_i'^2}}{\partial x_j}\\&+2\delta_{i3}\overline{u_i'\left(\frac{\theta'}{\bar{\theta}}\right)}g+2f_c\varepsilon_{ij3}\overline{u_i'u_j'}-\frac{2}{\bar{\rho}}\overline{u_i'\frac{\partial p'}{\partial x_i}}+2\nu\overline{u_i'\frac{\partial^2 u_i'}{\partial x_j^2}}\end{aligned}}$$

! still summed!

→ sum: conservation equation for TKE! $(TKE=0.5\rho\overline{u_{ii}'^2})$

→ 3 conservation equations for velocity variances....

Higher order moments: velocity variances

Have 1 equation, summed (i=1,2,3)

$$\frac{\partial \overline{u_i'^2}}{\partial t} + \overline{u}_j \frac{\partial \overline{u_i'^2}}{\partial x_j} = ... \longrightarrow \frac{\partial \overline{u_1'^2}}{\partial t} + \frac{\partial \overline{u_2'^2}}{\partial t} + \frac{\partial \overline{u_3'^2}}{\partial t} + ...$$

$$\left.\begin{array}{l}\text{all terms with } i=1: =A \\ \text{all terms with } i=2: =B \\ \text{all terms with } i=3: =C\end{array}\right\} \quad A+B+C=0$$

$$A+B+C=0 \xleftarrow{\text{consistent}} \begin{array}{l}A=0 \\ B=0 \\ C=0\end{array}$$

Higher order moments: velocity variances

3 conservation equations for velocity variances

$\boxed{u_1:}$

$$\frac{\partial \overline{u'^2_1}}{\partial t} + \overline{u}_j \frac{\partial \overline{u'^2_1}}{\partial x_j} = -2\overline{u'_1 u'_j}\frac{\partial \overline{u}_1}{\partial x_j} - \frac{\partial \overline{u'_j u'^2_1}}{\partial x_j}$$

$$+2f_c\overline{u'_1 u'_2} - \frac{2}{\overline{\rho}}\overline{\frac{\partial u'_1 p'}{\partial x_1}} + \frac{2}{\overline{\rho}}\overline{p'\frac{\partial u'_1}{\partial x_1}} - 2\nu \overline{u'_1 \frac{\partial^2 u'_1}{\partial x_1^2}}$$

$\boxed{u_2:}$

$$\frac{\partial \overline{u'^2_2}}{\partial t} + \overline{u}_j \frac{\partial \overline{u'^2_2}}{\partial x_j} = -2\overline{u'_2 u'_j}\frac{\partial \overline{u}_2}{\partial x_j} - \frac{\partial \overline{u'_j u'^2_2}}{\partial x_j}$$

$$-2f_c\overline{u'_1 u'_2} - \frac{2}{\overline{\rho}}\overline{\frac{\partial u'_2 p'}{\partial x_2}} + \frac{2}{\overline{\rho}}\overline{p'\frac{\partial u'_2}{\partial x_2}} - 2\nu \overline{u'_2 \frac{\partial^2 u'_2}{\partial x_2^2}}$$

$\boxed{u_3:}$

$$\frac{\partial \overline{u'^2_3}}{\partial t} + \overline{u}_j \frac{\partial \overline{u'^2_3}}{\partial x_j} = -2\overline{u'_3 u'_j}\frac{\partial \overline{u}_3}{\partial x_j} - \frac{\partial \overline{u'_j u'^2_3}}{\partial x_j}$$

$$+\frac{2g}{\overline{\theta}}\overline{u'_3\theta'} - 2\overline{u'_3 u'_j}\frac{\partial \overline{u}_3}{\partial x_j} - \frac{2}{\overline{\rho}}\overline{\frac{\partial u'_3 p'}{\partial x_3}} + \frac{2}{\overline{\rho}}\overline{p'\frac{\partial u'_3}{\partial x_3}} - 2\nu \overline{u'_3 \frac{\partial^2 u'_3}{\partial x_3^2}}$$

Higher order moments

→ turbulence: ‘introduced’ through Reynolds decomposition
→ have eq. for **mean turbulent flow**
→ even for higher moments….

Closure problem

conservation equation for mean flow:

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\overline{u_i' u_j'}\right) = -\delta_{i3} g + f_c \varepsilon_{ij3} \bar{u}_j - \frac{1}{\bar{\rho}} \frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j^2}$$

→ 2nd moments appear! (new variables)

→ conservation equation for new variable…

$$\frac{\partial \overline{u_i'^2}}{\partial t} + \bar{u}_j \frac{\partial \overline{u_i'^2}}{\partial x_j} = -2\overline{u_i' u_j'} \frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \overline{u_j' u_i'^2}}{\partial x_j}$$

→ 3rd moments appear!

$$+2\delta_{i3} \overline{u_i' \left(\frac{\theta'}{\bar{\theta}}\right)} g + 2 f_c \varepsilon_{ij3} \overline{u_i' u_j'} - \frac{2}{\bar{\rho}} \overline{u_i' \frac{\partial p'}{\partial x_i}} + 2\nu \overline{u_i' \frac{\partial^2 u_i'}{\partial x_j^2}}$$

Closure problem

Before:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\delta_{i3} g + f_c \varepsilon_{ij3} u_j - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial \sigma_{ij}}{\partial x_j}$$

→ no turbulence

→ 7 variables (u, v, w, p, q, θ, ρ)

→ 7 equations

After:

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\overline{u_i' u_j'}\right) = -\delta_{i3} g + f_c \varepsilon_{ij3} \overline{u}_j - \frac{1}{\overline{\rho}}\frac{\partial \overline{p}}{\partial x_i} + \nu \frac{\partial^2 \overline{u}_i}{\partial x_j^2}$$

→ turbulence introduced

→ 12 new variables (auto-)covariances

→ **conservation equations for covariances:**

  → even more new variables…

$\boxed{\text{Closure problem!}}$

Dilema

Description of a turbulent flow:

1) must: resolve millimeters / seconds

→ cpu-problem

2) Reynolds decomposition / averaging

→ closure problem

way out:

→ turbulence closure!

Turbulence Closure

→ at some point: break the vicious circle
'new equations → even more new variables'!

for example at first order:

→ **parameterize** the unknown 2nd moments

→ $\overline{u'w'} = 0$, or $\overline{u'w'}=$const. ...

→ or on the basis of similarity theory

→ ...

Turbulence Closure

Closure of order N:

- conservation equations up to order N
- $(N+1)^{th}$ moments parameterized

$$M_{n+1}^{j_{n+1}} = f(M_n^{j_n},\ M_{n-1}^{j_{n-1}},\ \ldots M_1^{j_1},\ P_{j_{n+1}})$$

function f: parameterization

→ dependent on the moments up to N^{th} order

→ need not to consider all

→ parameters P_j

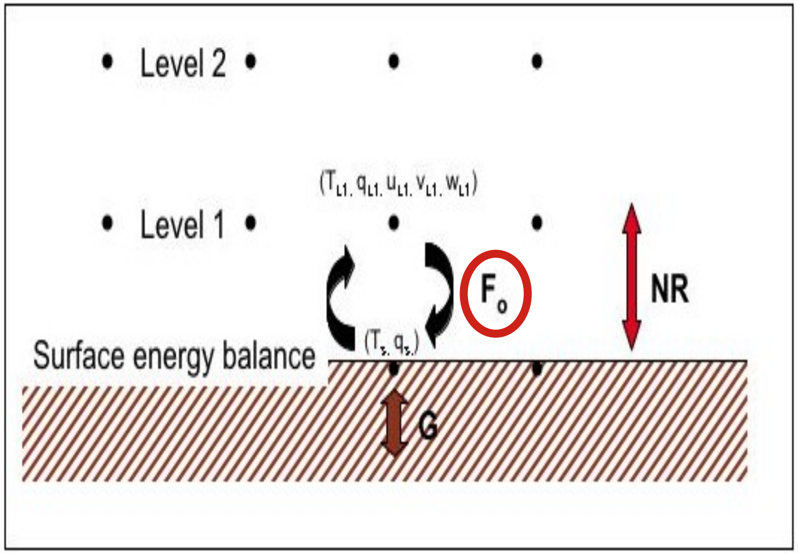
Turbulence Closure

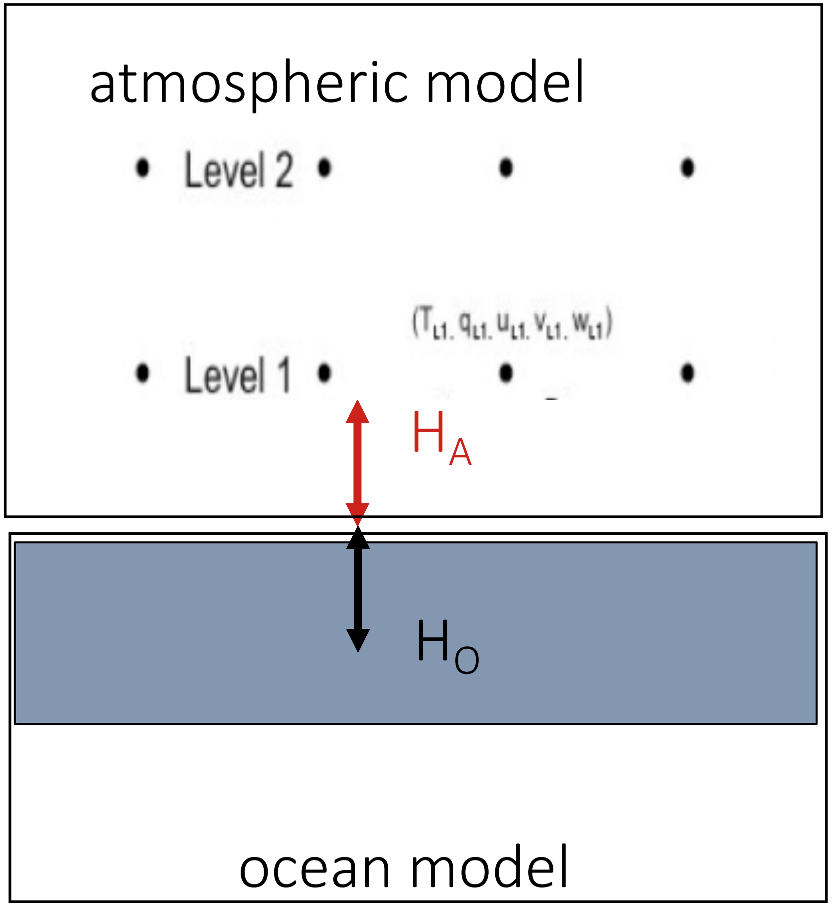
Parameterization:

- consistent with the unknown in terms of
  → units
  → tensor symmetries
- invariant with respect to transformations
  → e.g. coordinate transformation
- no violation of overarching principles
  → e.g. energy conservation

Overarching principle: energy conservation

example: climate model





both, H_A and H_O 'optimal'
→ let: +/- 1 Wm^{-2} (!)
→ integration over 100 years.....
→ energy conservation of the complete system!

First order closure

First order closure

→ conservation equations up to mean variables (1st moments)
→ parameterization (co-) variances (2nd moments)

approach:

remember....

$$\sigma_{ij}^{mol} = \rho \nu s_{ij} =: \rho \nu \left(\partial u_i / \partial x_j + \partial u_j / \partial x_i \right)$$

shear stress tensor proportional to deformation rate

... in analogy:

First order closure

… in analogy:

$$\frac{\tau_{ij}}{\overline{\rho}} = -(K_m)_{ij}\left(\partial\overline{u}_i / \partial x_j + \partial\overline{u}_j / \partial x_i\right)$$

turbulent shear stress tensor
proportional
to **mean** deformation rate

Reynolds stress tensor

factor of proportionality: ‘K’
→ **‘K theory’**

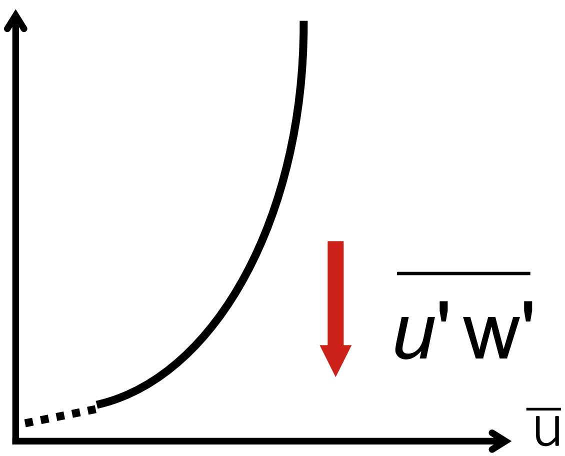
K Theory

$$\frac{\tau_{ij}}{\bar{\rho}} = -(K_m)_{ij}\left(\partial\bar{u}_i / \partial x_j + \partial\bar{u}_j / \partial x_i\right)$$

proportionality factor: ‘K’

→ ‘K-Theory’

→ minus sign: transport ‘against’ the gradients



$(K_m)_{ij}$: ‘eddy viscosity’

$$\overline{u_1'u_3'} = \frac{\tau_{13}}{\bar{\rho}} = -(K_m)_{13}\left(\partial\bar{u}_1 / \partial x_3 + \partial\bar{u}_3 / \partial x_1\right)$$

Intermediate summary

Reynolds decomposition and averaging:
→ additional terms in the budget equations (new variables!)
→ in each: flux divergence terms

Dilemma
→ if turbulence to be resolved: need (Super Computer)^3
→ if Reynolds decomposition and averaging: closure problem

Closure problem: the higher order conservation equations one works out, the larger is the number of new unknowns introduced

Closure of order n: consider budget eqns up to order n
→ parameterize moments of order n+1

Intermediate summary

1st order closure:

→ equations up to 1 (i.e., budget eqns for mean variables)

→ parameterize the 2nd order moments

→ i.e., $\overline{u'w'}, \ \overline{w'\theta'}, \ \overline{w'q'}, \ ..$

1st order closure, example:

energy equation:

$$\frac{\partial\overline{\theta}}{\partial t} + \overline{u}_j \frac{\partial\overline{\theta}}{\partial x_j} + \frac{\partial}{\partial x_j}(\overline{u_j'\theta'}) = \nu_\theta \frac{\partial^2\overline{\theta}}{\partial x_j^2} + \frac{1}{\overline{\rho} c_p}\overline{QR}$$

→ K-closure: $\overline{w'\theta'} = -K_H \frac{\partial\overline{\theta}}{\partial x_3}$

K Theory

If coordinate system || mean wind,
and horizontally homogenous:

$$\frac{\tau_{13}}{\bar{\rho}} = \overline{u'_1 u'_3} = -(K_m)_{13} \frac{\partial \bar{u}_1}{\partial x_3}$$

→ K_m instead of $(K_m)_{13}$

general (variable a):

$$\overline{w'a'} = -K_{a3} \frac{\partial \bar{a}}{\partial x_3}$$

$$\left(\overline{v'a'} = -K_{a2} \frac{\partial \bar{a}}{\partial x_2}, \quad \overline{u'a'} = -K_{a1} \frac{\partial \bar{a}}{\partial x_1} \right)$$

Everything OK?

→ problem shifted!
→ need K instead of 2nd order moment

K Theory

Determination of K:

- K= const. (old models) ‘not the best solution…’ (Stull 1988)

- K from Prandtl/ v. Kàrmàn theory just neutral….

- K from similarity theory (exercise)

Formulations for K_m

Table 6-4. Examples of parameterizations for the eddy viscosity, K, in the boundary layer.

Neutral Surface Layer:

$K = \text{constant}$	not the best parameterization
$K = u_*^2 T_o$	where u_* is the friction velocity
$K = U^2 T_o$	where T_o is a timescale
$K = k z u_*$	where k is von Karman's constant
$K = k^2 z^2 [(\partial\bar{U}/\partial z)^2 + (\partial\bar{V}/\partial z)^2]^{1/2}$	from mixing-length theory
$K = l^2 (\partial\bar{U}/\partial z)^2$	where $l = k(z+z_o)/\{1+[k(z+z_o)/\Lambda]\}$, Λ=length scale

Diabatic Surface Layer (generally, $K_{\text{statically unstable}} > K_{\text{neutral}} > K_{\text{statically stable}}$)

$K = k z u_* / \phi_M(z/L)$	where ϕ_M a dimensionless shear (see appendix A), and L is the Obukhov length (appendix A)
$K = k^2 z^2 [(\partial\bar{U}/\partial z) + \{(g/\bar{\theta}_v)\cdot\vert\partial\bar{\theta}_v/\partial z\vert\}^{1/2}]$	for statically unstable conditions
$K = k^2 z^2 [(\partial\bar{U}/\partial z) - (L_*/z)^{1/6}\{(15g/\bar{\theta}_v)\cdot\vert\partial\bar{\theta}_v/\partial z\vert\}^{1/2}]$	for statically stable conditions, where $L_* = -\theta u_*^2/(15 k g \theta_*)$

Neutral or Stable Boundary Layer

$K = \text{constant}$	see Ekman Spiral derivation in next subsection
$K = K(h) + [(h-z)/(h-z_{SL})]^2 \{K(z_{SL}) - K(h) + (z-z_{SL})[\partial K/\partial z\vert_{z_{SL}} + 2(K(z_{SL})-K(h))/(h-z_{SL})]\}$	this is known as the O'Brien cubic polynomial approximation (O'Brien, 1970), see Fig 6-2, where z_{SL} represents the surface layer depth.

Unstable (Convective) Boundary Layer:

$K = 1.1\,[(R_c - Ri)\, l^2 / Ri\,]\,\vert\partial\bar{U}/\partial z\vert$	for $\partial\bar{\theta}_v/\partial z > 0$	where $l = kz$ for $z < 200$ m and
$K = (1 - 18\,Ri)^{-1/2}\, l^2\, \vert\partial\bar{U}/\partial z\vert$	for $\partial\bar{\theta}_v/\partial z < 0$	$l = 70$ m for $z > 200$ m.

Numerical Model Approximation for Anelastic 3-D Flow:

$K = (0.25\,\Delta)^2 \cdot \vert 0.5\, \Sigma_i \Sigma_j\, [\partial\bar{U}_i/\partial x_j + \partial\bar{U}_j/\partial x_i - (2/3)\delta_{ij}\Sigma_k(\partial\bar{U}_k/\partial x_k)]^2\, \vert^{1/2}$ where Δ=grid size

Stull (1988)

K Theory

The important K's:

$$\overline{u'w'} = -K_m \frac{\partial \overline{u}}{\partial x_3}$$

momentum → 'eddy viscosity'

$$\overline{w'\theta'} = -K_H \frac{\partial \overline{\theta}}{\partial x_3}$$

sensible heat → 'eddy conductivity'

$$\overline{w'q'} = -K_q \frac{\partial \overline{q}}{\partial x_3}$$

latent heat

$$\overline{w'C'} = -K_C \frac{\partial \overline{C}}{\partial x_3}$$

tracers → 'eddy diffusivity'

K Theory

Properties of the K's:

$$K_m \neq K_H \approx K_q \approx K_C$$

relatively well known
(easy to determine)

more difficult
to determine

only in recent
years

$$\overline{w'C'} = -K_c \frac{\partial \bar{C}}{\partial x_3} \quad \rightarrow K_c = -\overline{w'C'} \,/\, \frac{\partial \bar{C}}{\partial x_3}$$

→ need to measure trubulent fluctuations, C'

→ proton transfer reaction time-of-flight mass spectrometer, PTR ToF-MS

→ Innsbruck Atmospheric Observatory, IAO

K Theory

Properties of the K's:

$$K_m \neq K_H \approx K_q \approx K_C$$

relatively well known (easy to determine)

more difficult to determine

only in recent years

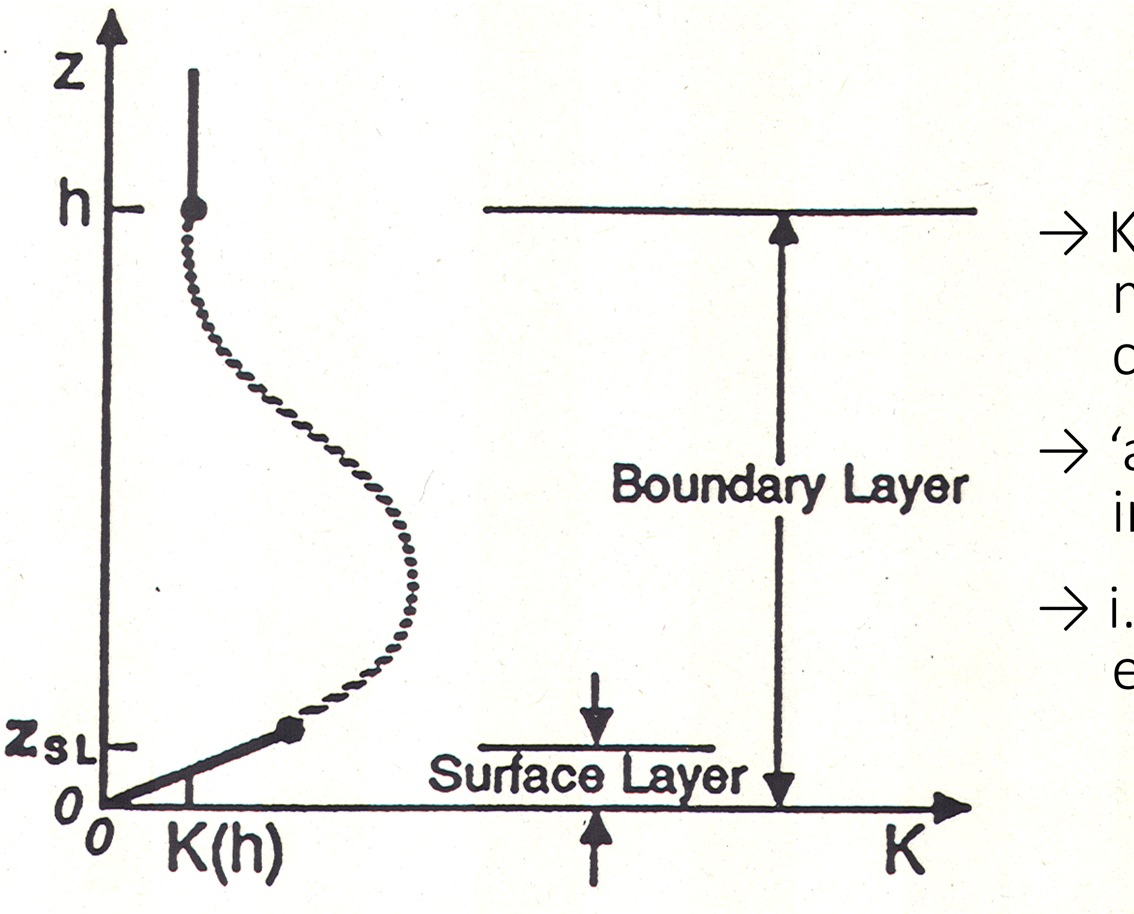
→ K_m most closely to 'external variable'
→ K_H modifies stability

$$K_{unstable} > K_{neutral} > K_{stable}$$

- for all variables
- efficiency of exchange

K Theory

Height dependence (from similarity theory)



→ K-theory often in models (esp. global climate)

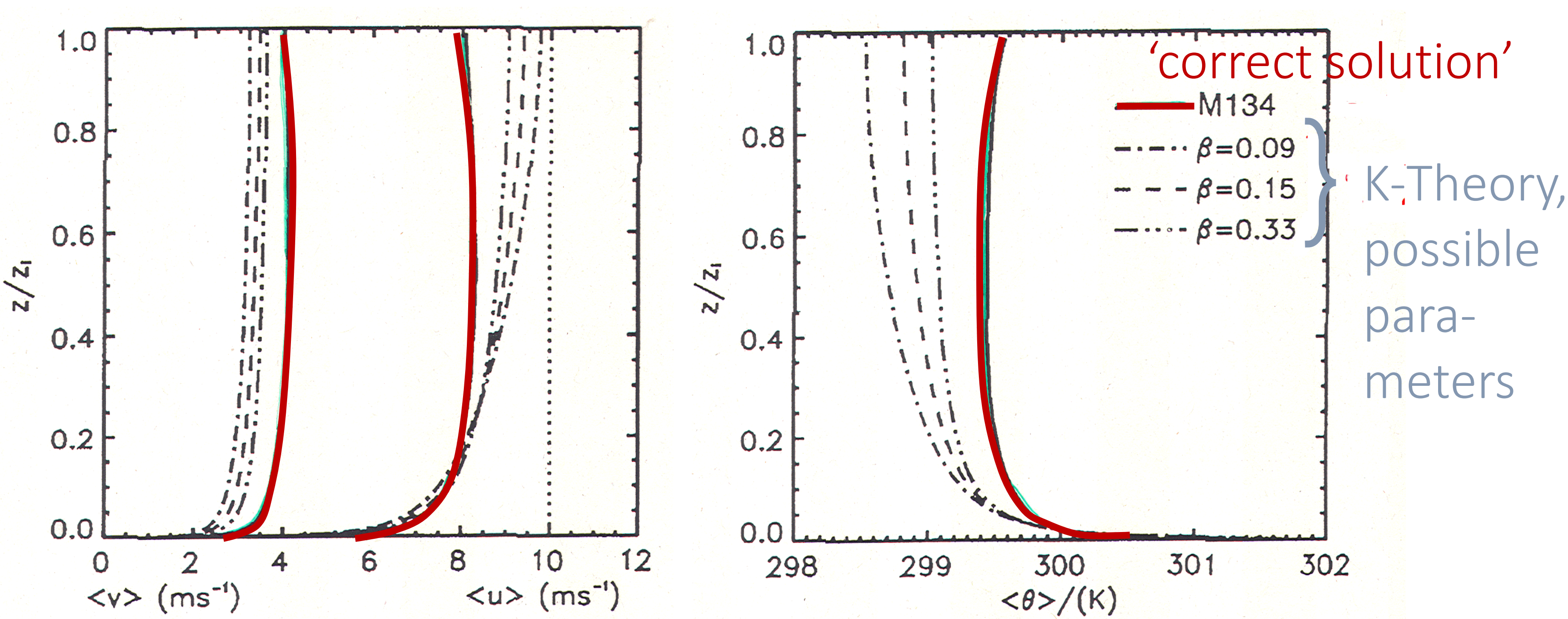
→ ‘all turbulence information in K’

→ i.e. turbulence entirely parameterized

K Theory

→ limitation of K-Theory:

→ under certain conditions correct solution not attainable



1.5 order closure

1.5-order Closure

→ first order: turbulence entirely in K…
→ cannot reproduce fast temporal changes

→ 2nd order…. (laborious, lots of new equations)
→ compromise:

- conservation equation for TKE
- conservation equation for dissipation rate ε

still: $$\overline{u'w'} = -K_m \frac{\partial \overline{u}}{\partial x_3}$$ but: $$K_m \propto \frac{\overline{e}^2}{\varepsilon}$$

1.5-order Closure

still: $$\overline{u'w'} = -K_m \frac{\partial \overline{u}}{\partial x_3}$$

but: $$K_m \propto \frac{\overline{e}^2}{\varepsilon}$$

→ ‘few’ higher order moments considered

$$TKE = \frac{1}{2}\rho\overline{u_{ii}'^2}, \quad e = TKE / \rho$$

→ still: ‘flux proportional mean gradients’

→ order: 1.5

Non-local Closure

→ Up to now: all closures ‘local’

→ local= flux **of the height z** parameterized with
the gradient of **height z**

→ more general: moment of order N+1 at location is $\vec{r}$
parameterized using moments of order N at location $\vec{r}$

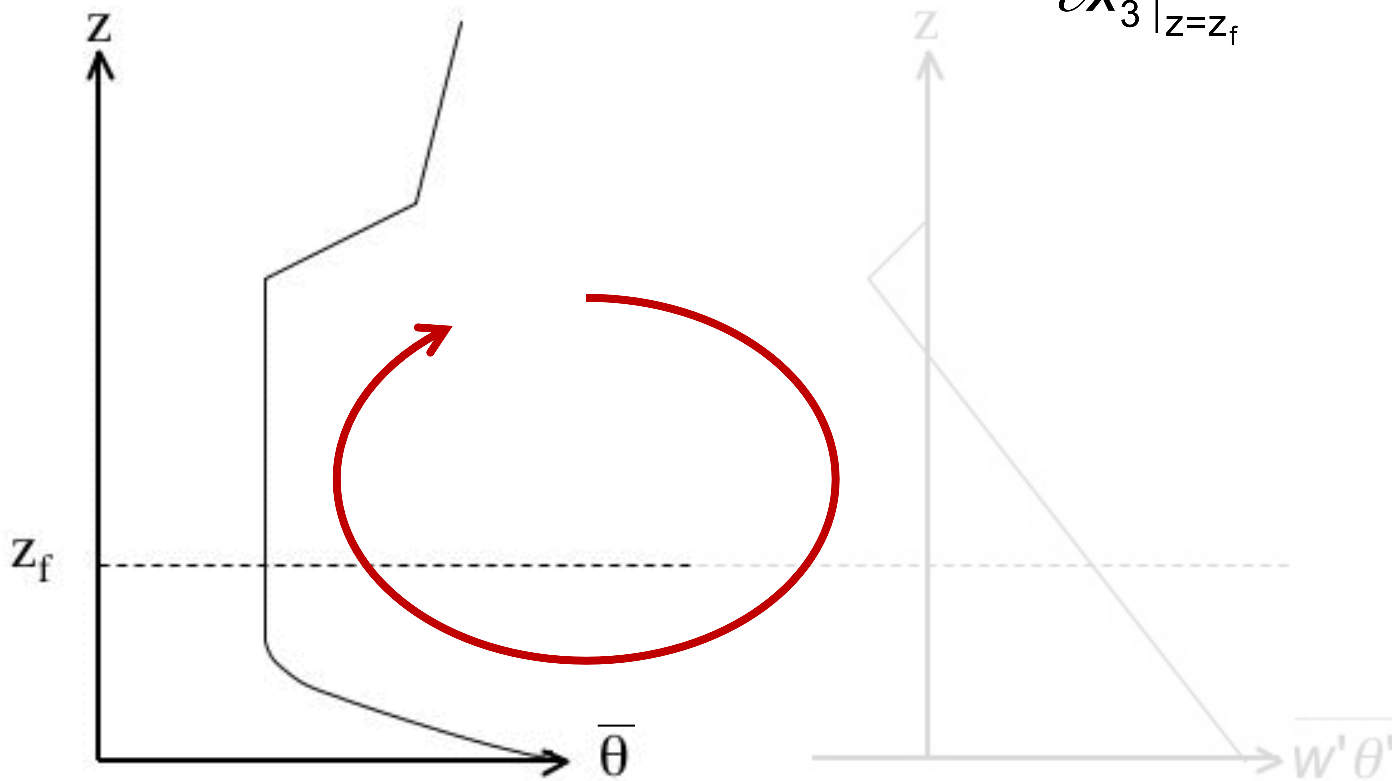
→ this is not always fortunate .....

→ example: convective boundary layer

Non-local Closure

CBL:

$$\overline{w'\theta'}(z_f) = -K_H \left.\frac{\partial\overline{\theta}}{\partial x_3}\right|_{z=z_f} \Rightarrow K_H = ?$$



→ CBL: exchange through large eddies [$O(z_i)$]

Non-local Closure

CBL: pragmatic approach:
→ 'counter gradient term'

$$\overline{w'\theta'}(z_f) = -K_H\left(\frac{\partial\overline{\theta}}{\partial x_3} - \gamma_\theta\right)$$

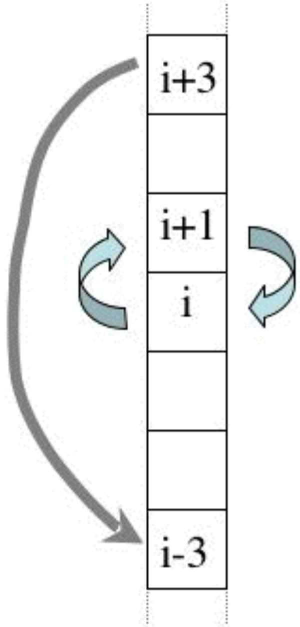
γ_θ from conservation equation for θ

→ general: non-local closure:

→ Transilient Turbulence Theory (Stull)

Transilient Turbulence Theory

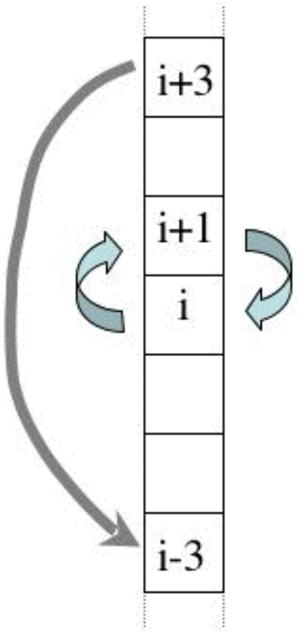
Principle: Moments to be parameterized for each 'grid cell': influenced by variables at all possible other grid points (non-local!)



<u>'concentration' C:</u>

$$\overline{C}_i(t+\Delta t) = \sum_{j=1}^{n} M_{ij}(t,\Delta t)\cdot\overline{C}_j(t)$$

Transilient Turbulence Theory



'concentration' C:

$$\overline{C}_i(t+\Delta t) = \sum_{j=1}^{n} M_{ij}(t,\Delta t)\cdot\overline{C}_j(t)$$

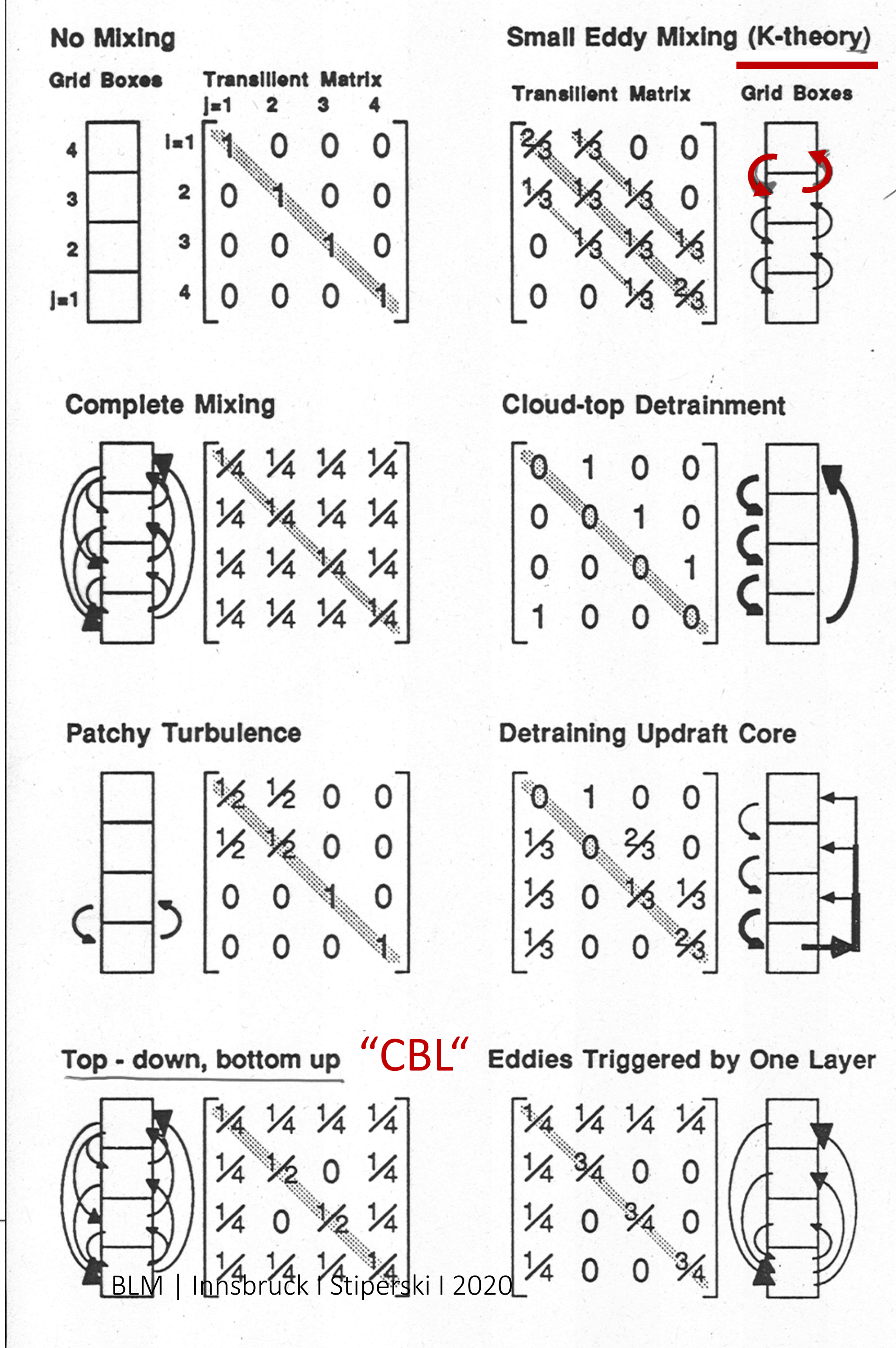
flux at box i:

$$\overline{w'C'}(i) = \left(\frac{\Delta z}{\Delta t}\right)\sum_{k=1}^{i}\sum_{j=1}^{n} M_{ij}(\overline{C}_i - \overline{C}_j)$$

→ combination of the 'exchange' of any box with all the others

→ determination of M_{ij}???

Transilient Turbulence Theory



Which closure?

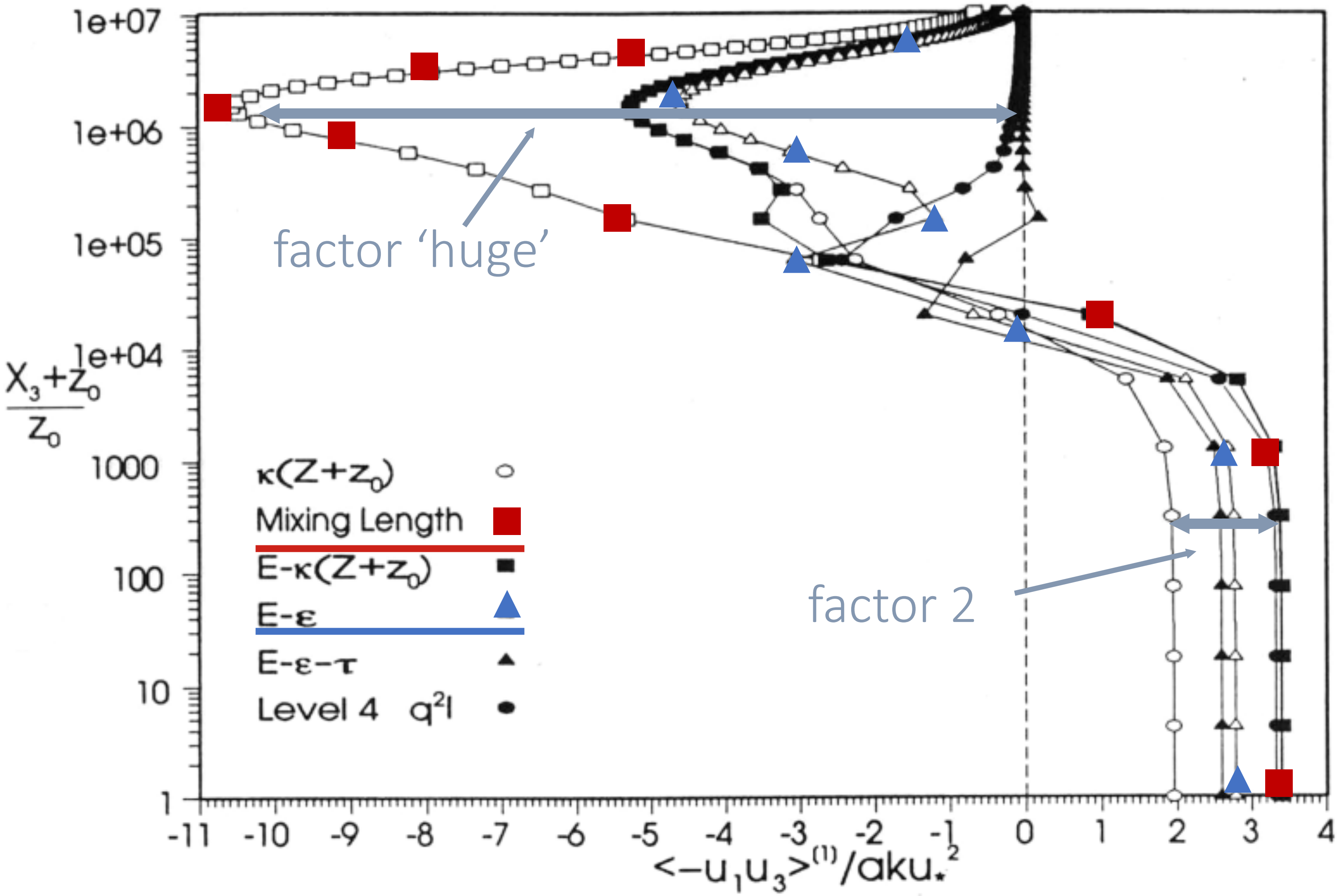
The higher order the closure, the better?

- 13rd order….
  → equations up to the 13th moment
  → 14th moment parameterized….
  → no idea about these moments!
  → statistically highly uncertain!

→ practice:

- many atmospheric models: 1st order!
- elaborate models: 1.5 or 2nd order
  → operational NWP (WRF, COSMO, Arome, .. : 1.5 order

Which closure?



Which closure?

→ small-scale turbulence:
local closure ok

SL,
neutral PBL
stable PBL

→ large eddies (CBL):
at least 1.5 or 2nd order

CBL
canopies

alternative: Large Eddy Simulation

Large eddy simulation (LES)

Reynolds averaged models:

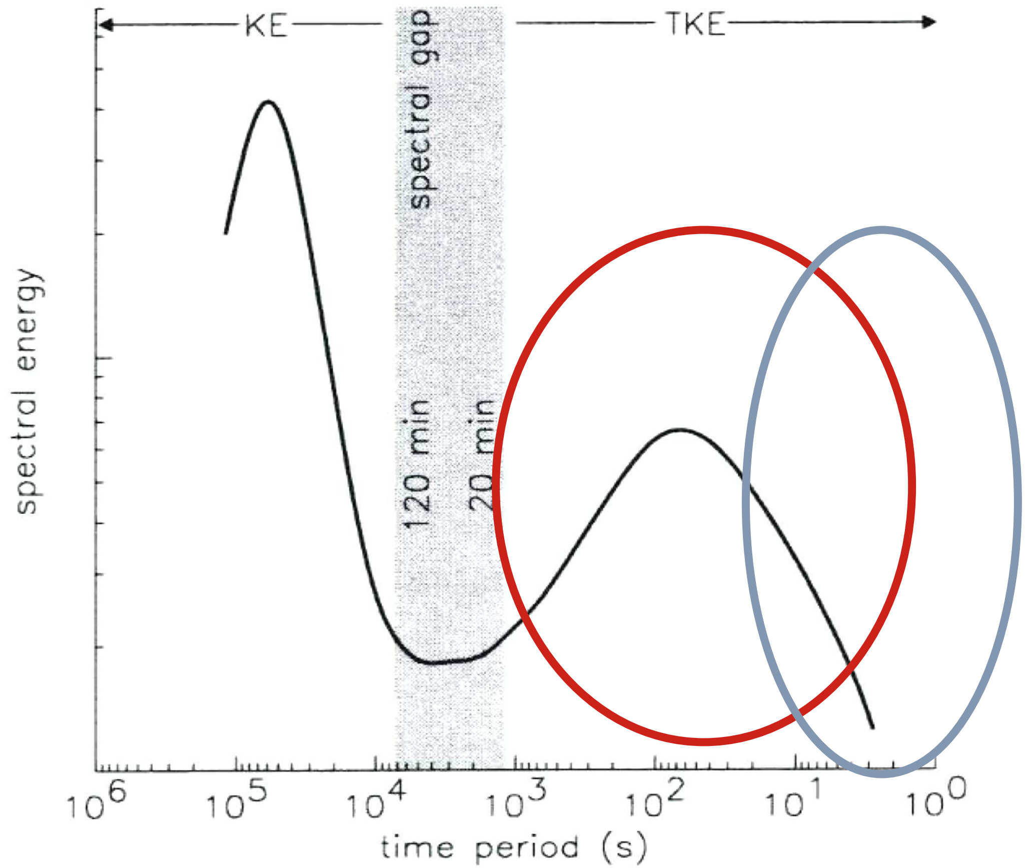
→ entire turbulence spectrum parameterized

Large Eddy Simulation

→ filtered equations

→ large eddies resolved

→ only small ones parameterized



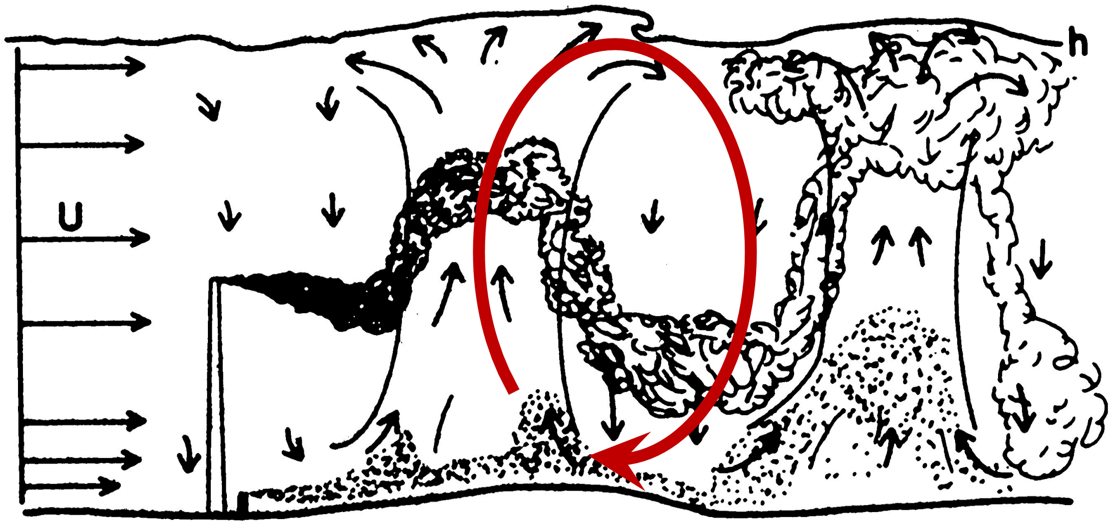
Large eddy simulation (LES)

LES:

→ resolves big eddies

→ turbulence model dependent on Δx!

→ highly CPU-intensive!



Direct Numerical Simulations (DNS)

Summary: Closure

→ fundamental problem
→ closure approach necessary

→ local closures (K-theory, e - ε, ..)
→ non-local closures

→ expensive alternative: LES
→ DNS

Summary

Turbulence in the conservation equations
→ new terms: higher moments
→ flux divergence

- Closure problem!
  → closure: order N
  → often 1. order (K-theory)
  → local and non-local closures (CBL!)
- each numerical model needs a closure
  → NWP model, climate model, (but also all the others…)
  → the closure is the turbulence model
  → it describes the earth-atmosphere interaction
- Idealized solution: Ekman Spiral
  → Chapter A….