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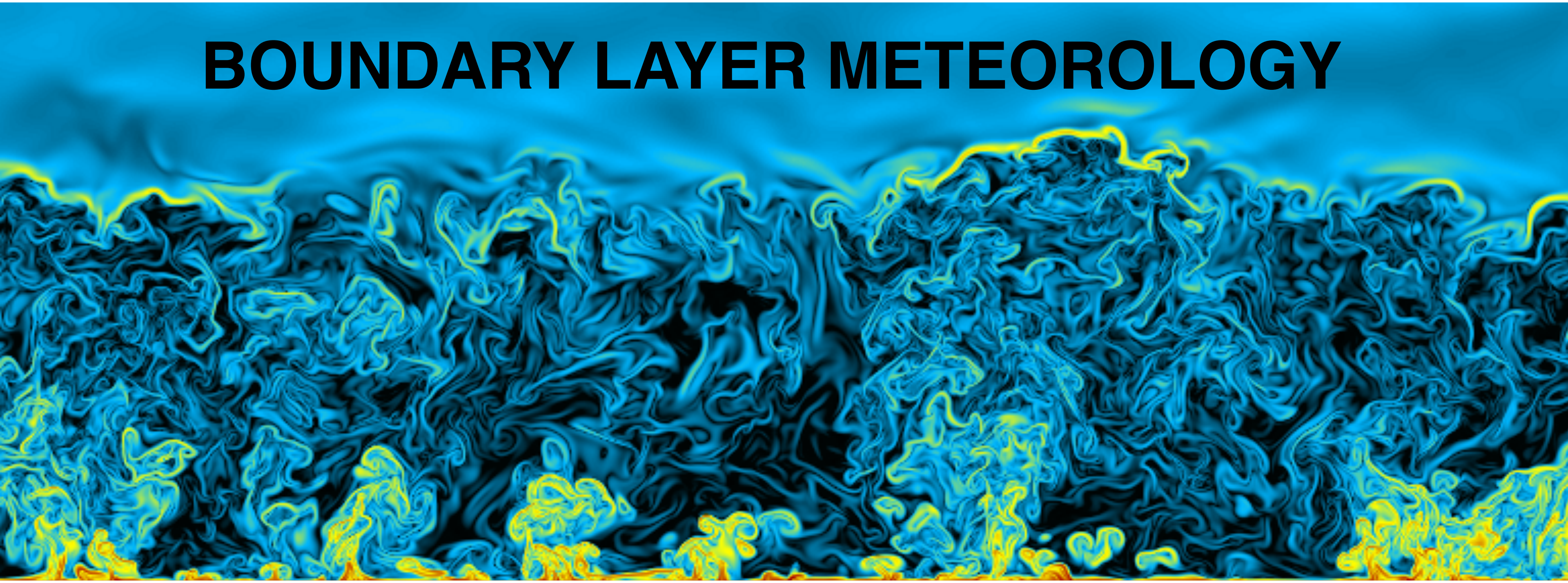

BOUNDARY LAYER METEOROLOGY

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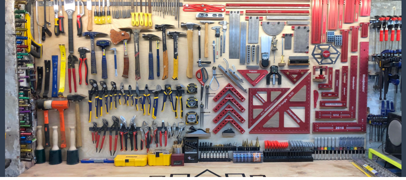
Chapter 3

Statistical treatment of turbulence

Revision



Pdf's

→ probability density function to describe turbulent variables

→ fully characterized through its *moments (variance, skewness..)*

Stationarity

→ all moments do not change with time

→ in practice: up to second moments enough

Homogeneity

→ is stationarity in space (horizontal!)

Averaging

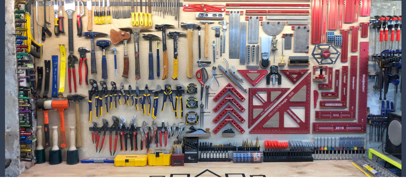
→ average over all possible realizations→ ensemble average

Reynold's averaging

→ separate the turbulence and the non-turbulent motions

→ $\mathrm{u} = \bar{u} + u'$

Revision



Ergodic Hypothesis

→" time ave. of stationary rand. var. and space ave. of homogeneous rand. var. converge to ensemble ave. over all realizations"

→ time/space average → ens. av.

Taylor Hypothesis

→ "Turbulence is frozen during the time it travels across instrument"

Time average

→ Point measurements (turbulence towers) – average over which time?

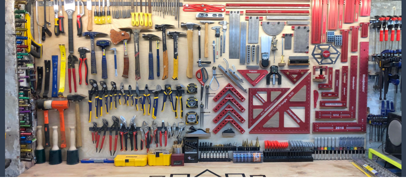
Space average

→ Distributed measurements

→ Lagrangean platforms

→ Volume averaged measurements

Revision



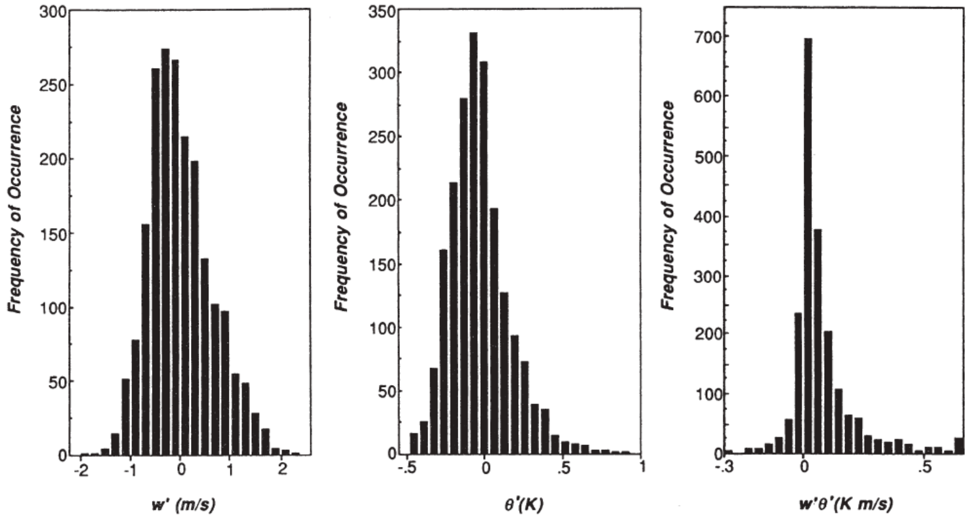
Covariances and Transport

→ Non-linear turbulence product of two variables

→ Non-zero if two signals are correlated (arise from same process)

→ Represent turbulent fluxes (transport)

→ Quadrant analysis: process



Stull (1988)

Turbulent fluxes

- kinematic fluxes: $\overline{w'\theta'}, \overline{w'q'}, \overline{u'w'}, (\overline{v'w'}), \overline{w'c'}$

- In energy units:

<u>turbulent transport of</u>

$\rho c_p \overline{w'\theta'} =: H$ — sensible heat — $[H] = Wm^{-2}$

$\rho L_v \overline{w'q'} =: L_v E$ — latent heat — $[L_v E] = Wm^{-2}$

$\rho \overline{u'w'} =: M$ — momentum — $[M] = Nm^{-2}$

→ without average vertical velocity!

Turbulent fluxes

- flow description
  → conservation eq.

$$\frac{\partial \theta}{\partial t} + u_j \frac{\partial \theta}{\partial x_j} = \nu_\theta \frac{\partial^2 \theta}{\partial x_j^2} - \frac{1}{\rho c_p} \frac{\partial NR_j}{\partial x_j} - \frac{L_v E}{\rho c_p} + \frac{R_c}{\rho c_p}$$

for a turbulent flow:
→ Reynolds decomposition (no change, but fluctuations considered) of all variables
→ Reynolds average (whole equation)
→ Result: *conservation equation for mean flow, but turbulence considered*
→ in the terms, where we have products of two variables

- consider advection term

$$= u_1 \frac{\partial \theta}{\partial x_1} + u_2 \frac{\partial \theta}{\partial x_2} + u_3 \frac{\partial \theta}{\partial x_3}$$

Turbulent fluxes

- flow description
  → conservation eq.
- consider advection term

$$\frac{\partial \theta}{\partial t} + u_j \frac{\partial \theta}{\partial x_j} = \nu_\theta \frac{\partial^2 \theta}{\partial x_j^2} - \frac{1}{\rho c_p} \frac{\partial NR_j}{\partial x_j} - \frac{L_v E}{\rho c_p} + \frac{R_c}{\rho c_p}$$

$$= u_1 \frac{\partial \theta}{\partial x_1} + u_2 \frac{\partial \theta}{\partial x_2} + u_3 \frac{\partial \theta}{\partial x_3}$$

Reynolds decomposition:

$$= (\overline{u}_1 + u_1') \frac{\partial(\overline{\theta} + \theta')}{\partial x_1} + (\overline{u}_2 + u_2') \frac{\partial(\overline{\theta} + \theta')}{\partial x_2} + (\overline{u}_3 + u_3') \frac{\partial(\overline{\theta} + \theta')}{\partial x_3}$$

$$= \overline{u}_1 \frac{\partial \overline{\theta}}{\partial x_1} + \overline{u}_1 \frac{\partial \theta'}{\partial x_1} + u_1' \frac{\partial \overline{\theta}}{\partial x_1} + u_1' \frac{\partial \theta'}{\partial x_1}$$

$$+ \overline{u}_2 \frac{\partial \overline{\theta}}{\partial x_2} + \overline{u}_2 \frac{\partial \theta'}{\partial x_2} + u_2' \frac{\partial \overline{\theta}}{\partial x_2} + u_2' \frac{\partial \theta'}{\partial x_2}$$

$$+ \overline{u}_3 \frac{\partial \overline{\theta}}{\partial x_3} + \overline{u}_3 \frac{\partial \theta'}{\partial x_3} + u_3' \frac{\partial \overline{\theta}}{\partial x_3} + u_3' \frac{\partial \theta'}{\partial x_3}$$

Turbulent fluxes

Reynolds averaging:

$$\overline{(\overline{u}_1 + u_1')\frac{\partial(\overline{\theta}+\theta')}{\partial x_1} + (\overline{u}_2 + u_2')\frac{\partial(\overline{\theta}+\theta')}{\partial x_2} + (\overline{u}_3 + u_3')\frac{\partial(\overline{\theta}+\theta')}{\partial x_3}}$$

$$= \overline{\overline{u}_1\frac{\partial\overline{\theta}}{\partial x_1}} + \overline{\overline{u}_1\frac{\partial\theta'}{\partial x_1}} + \overline{u_1'\frac{\partial\overline{\theta}}{\partial x_1}} + \overline{u_1'\frac{\partial\theta'}{\partial x_1}}$$

$$+ \overline{\overline{u}_2\frac{\partial\overline{\theta}}{\partial x_2}} + \overline{\overline{u}_2\frac{\partial\theta'}{\partial x_2}} + \overline{u_2'\frac{\partial\overline{\theta}}{\partial x_2}} + \overline{u_2'\frac{\partial\theta'}{\partial x_2}}$$

$$+ \overline{\overline{u}_3\frac{\partial\overline{\theta}}{\partial x_3}} + \overline{\overline{u}_3\frac{\partial\theta'}{\partial x_3}} + \overline{u_3'\frac{\partial\overline{\theta}}{\partial x_3}} + \overline{u_3'\frac{\partial\theta'}{\partial x_3}}$$

0

$$\overline{a'\overline{b}} = 0$$

$$= \overline{u}_1\frac{\partial\overline{\theta}}{\partial x_1} + \overline{u_1'\frac{\partial\theta'}{\partial x_1}} + \overline{u}_2\frac{\partial\overline{\theta}}{\partial x_2} + \overline{u_2'\frac{\partial\theta'}{\partial x_2}} + \overline{u}_3\frac{\partial\overline{\theta}}{\partial x_3} + \overline{u_3'\frac{\partial\theta'}{\partial x_3}}$$

Turbulent fluxes

if:

→ horizontally homogeneous $\partial / \partial x_1 = \partial / \partial x_2 = 0$

→ mean vertical wind =0: $\overline{u}_3 = 0$

mean advection term:

$$= \overline{u}_1 \frac{\partial \overline{\theta}}{\partial x_1} + \overline{u_1' \frac{\partial \theta'}{\partial x_1}} + \overline{u}_2 \frac{\partial \overline{\theta}}{\partial x_2} + \overline{u_2' \frac{\partial \theta'}{\partial x_2}} + \overline{u}_3 \frac{\partial \overline{\theta}}{\partial x_3} + \overline{u_3' \frac{\partial \theta'}{\partial x_3}}$$

→ vertical advection term ≠ zero (even if horiz. homogeneous)

→ will see:

$$\overline{u_3' \frac{\partial \theta'}{\partial x_3}} = \frac{\partial}{\partial x_3} (\overline{u_3' \theta'})$$

Turbulent fluxes

before Reynolds treatment:

$$\frac{\partial \theta}{\partial t} + u_j \frac{\partial \theta}{\partial x_j} = \nu_\theta \frac{\partial^2 \theta}{\partial x_j^2} - \frac{1}{\rho c_p} \frac{\partial NR_j}{\partial x_j} - \frac{L_v E}{\rho c_p} + \frac{R_c}{\rho c_p}$$

after:

$$\frac{\partial \overline{\theta}}{\partial t} + u_j \frac{\partial \overline{\theta}}{\partial x_j} + \frac{\partial}{\partial x_3}\left(\overline{u_3'\theta'}\right) = \nu_\theta \frac{\partial^2 \overline{\theta}}{\partial x_j^2} - \frac{1}{\rho c_p} \frac{\partial \overline{NR_j}}{\partial x_j} - \frac{\overline{L_v E}}{\rho c_p} + \frac{\overline{R}_c}{\rho c_p}$$

→ here: horiz. homogeneous & no mean vertical velocity
→ additional term: flux divergence

Momentum transport

Special case:

- Co-variance between 2 velocity vectors

Tensor notation

$$\vec{b} \cdot \vec{a} = (b_1, b_2, b_3) \cdot (a_1, a_2, a_3)$$

$$= \begin{bmatrix} b_1 a_1 & b_1 a_2 & b_1 a_3 \\ b_2 a_1 & b_2 a_2 & b_2 a_3 \\ b_3 a_1 & b_3 a_2 & b_3 a_3 \end{bmatrix}$$

for velocity vector:

(u, v, w) instead of (u_1, u_2, u_3)

$$\vec{u} \cdot \vec{u} = (u, v, w) \cdot (u, v, w)$$

$$= \begin{pmatrix} uu & uv & uw \\ vu & vv & vw \\ wu & wv & ww \end{pmatrix}$$

Momentum transport

- co-variance between 2 velocity vectors

$$\overline{\text{cov}(u_i, u_j)} = \overline{u_i' u_j'} = \begin{pmatrix} \overline{u'u'} & \overline{u'v'} & \overline{u'w'} \\ \overline{v'u'} & \overline{v'v'} & \overline{v'w'} \\ \overline{w'u'} & \overline{w'v'} & \overline{w'w'} \end{pmatrix}$$

→ diagonal: variances
→ outside diagonal:

$\overline{u'w'}$ vertical transport of horizontal momentum

?

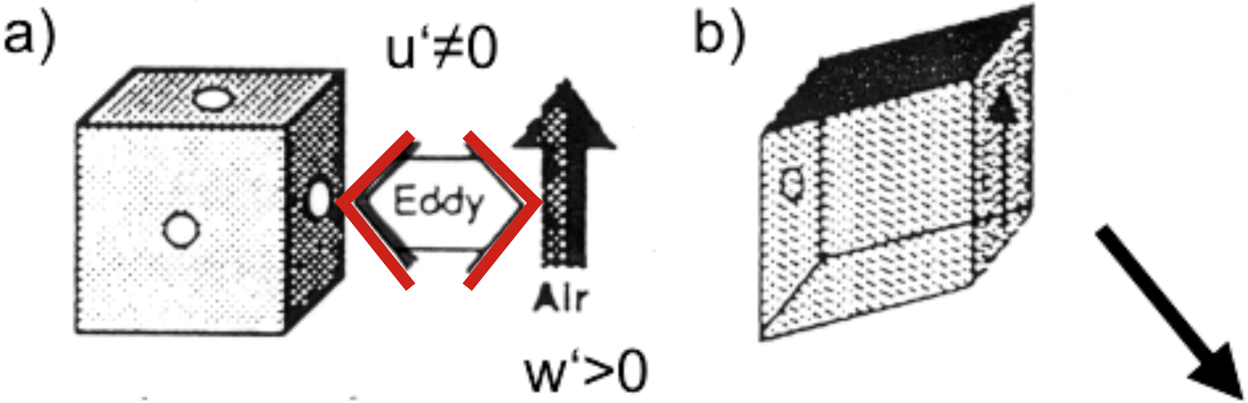
$\overline{w'u'}$ horizontal transport of vertical momentum

Momentum transport

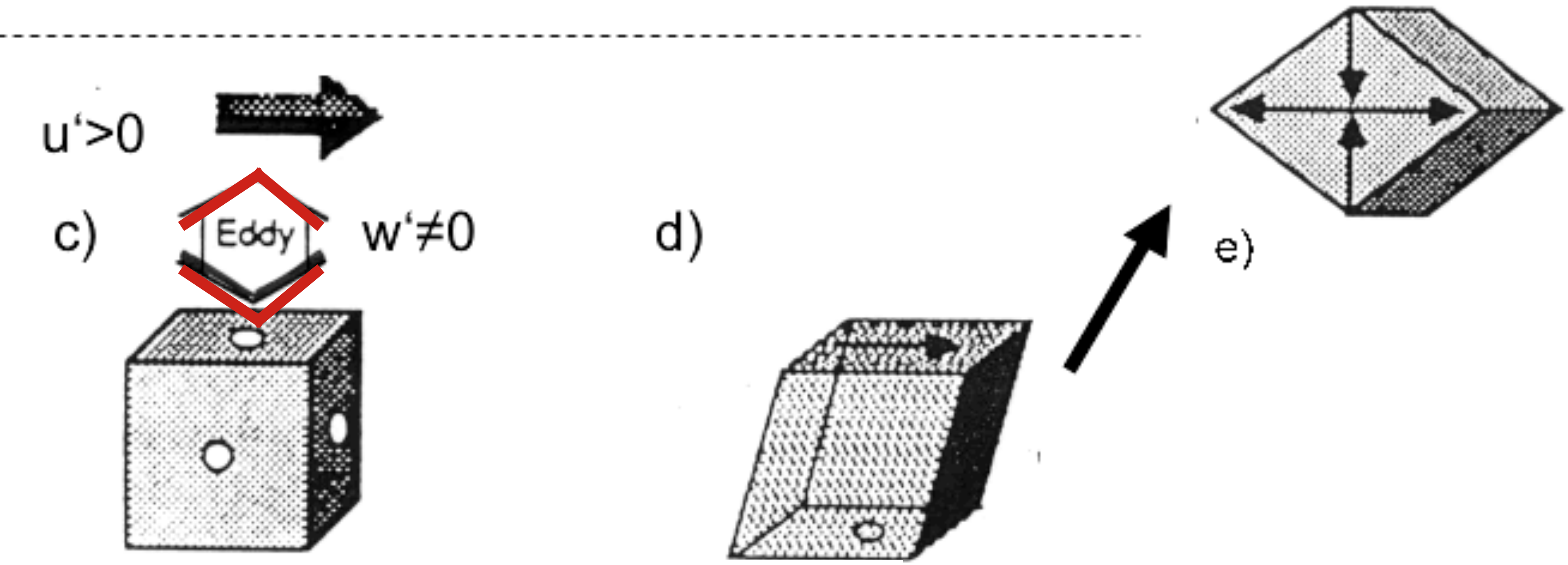
difference? $\overline{u'w'} \longleftrightarrow \overline{w'u'}$

effect:

consider deformation of fluid elements



→ w' transported to the cube

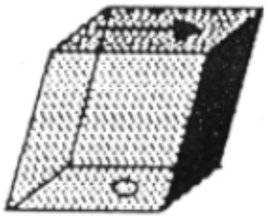


→ u' transported to the cube

Momentum transport

→ u’w’ and w’u’ indistinguishable
→ cov (u,w) **is symmetric**

process: friction
effect: deformation of fluid elements
→ force on one side of cube
→ units: pressure / shear stress



$$[\rho \overline{u'_i u'_j}] = Nm^{-2}$$

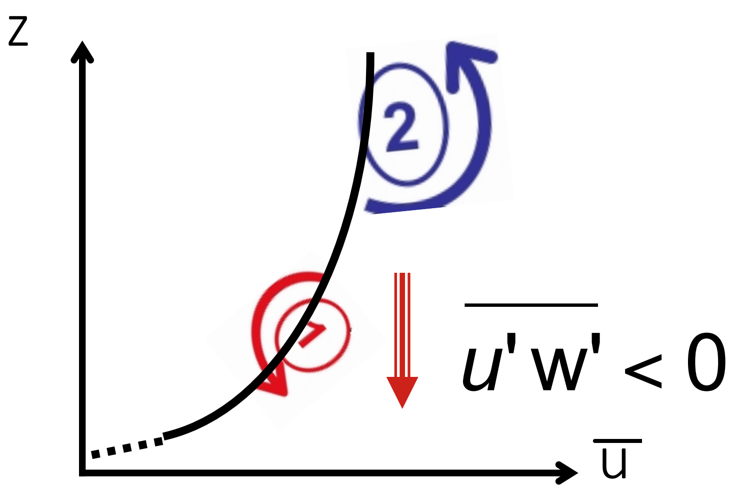
→ Reynolds stress tensor

Reynolds stress tensor

Def: $-\rho \overline{u'_i u'_j} =: \tau_{ij}$

process: friction

effect: deformation of fluid elements



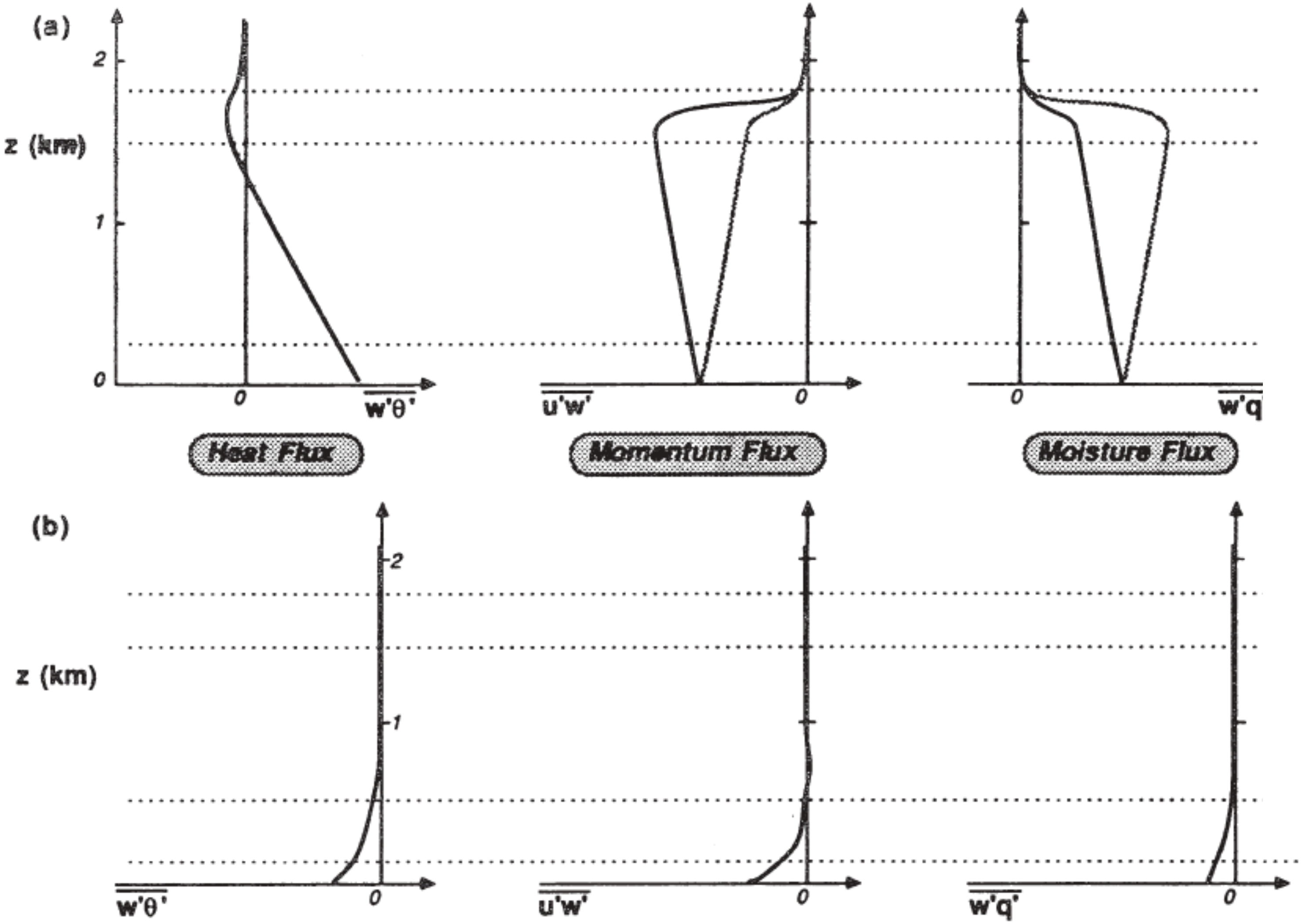
→ in particular: vertical transport

→ $\overline{u'w'}$, $\overline{v'w'}$

→ if coord. system parallel to mean wind:

→ $\overline{v'w'}=0$

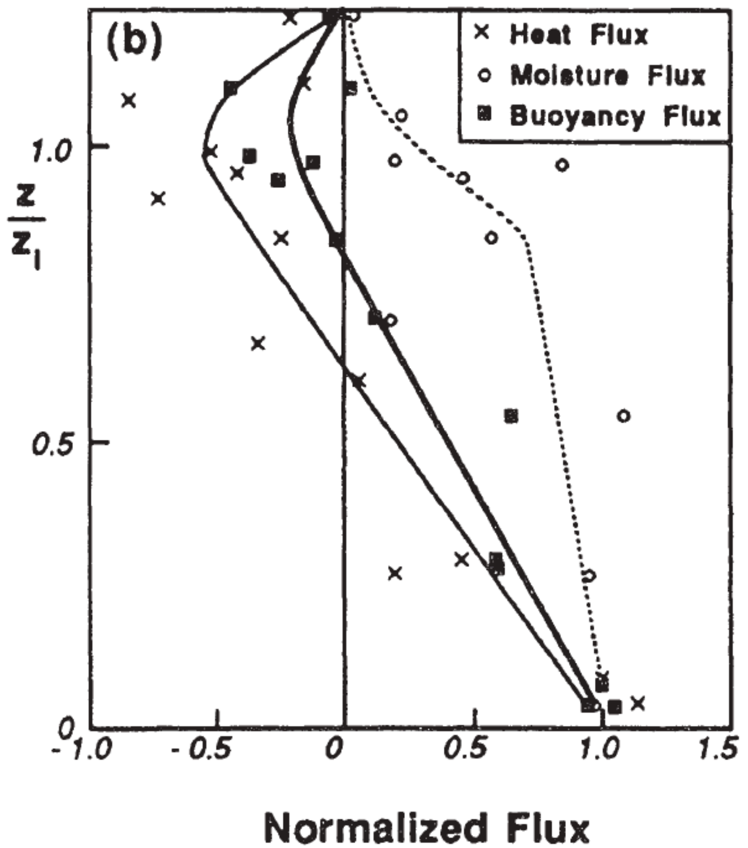
Typical profiles of fluxes



Stull (1988)

Typical profiles of heat fluxes

Why do we observe such a profile of heat flux?



Stull (1988)

Friction velocity

- near the ground:
  → friction (→ turbulence, mechanical)
  → shear stress, especially vertical
  → $\overline{u'w'}$, $\overline{v'w'}$
  → indicates: > how strong is deformation?
  > how much momentum transport for compensation?

- Def.: $u_* =: \left[(\overline{u'w'}_o)^2 + (\overline{v'w'}_o)^2 \right]^{1/4}$

characteristic velocity: *friction velocity*

Friction velocity

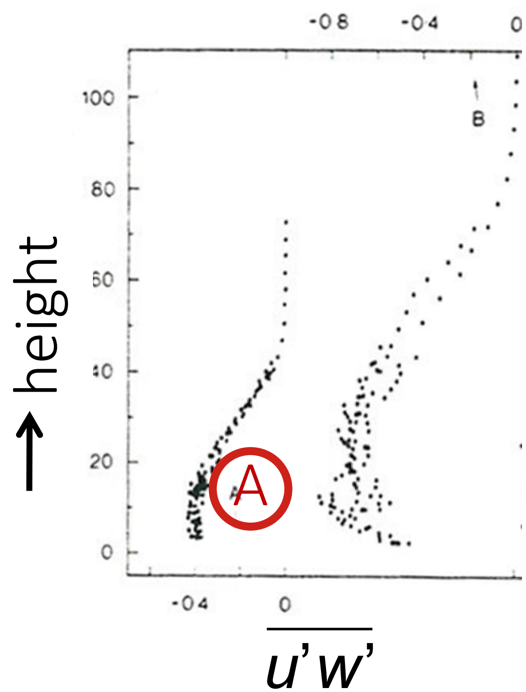
- defined based on *surface fluxes:*

$$u_* =: \left[(\overline{u'w'}_o)^2 + (\overline{v'w'}_o)^2 \right]^{1/4}$$

- fluxes approximately constant close to surface
- u_* characteristic velocity for *Surface Layer*

Surface layer

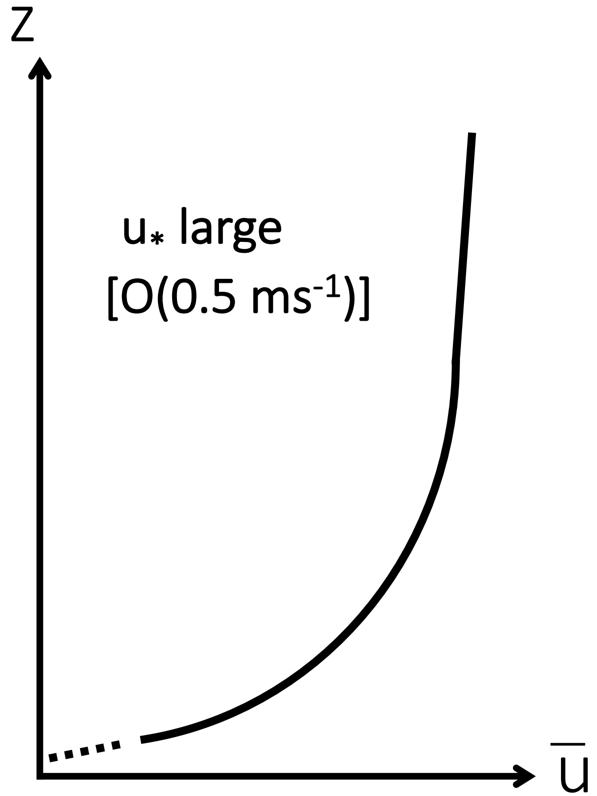
- Lowest 10% of PBL
- Fluxes change by 10%



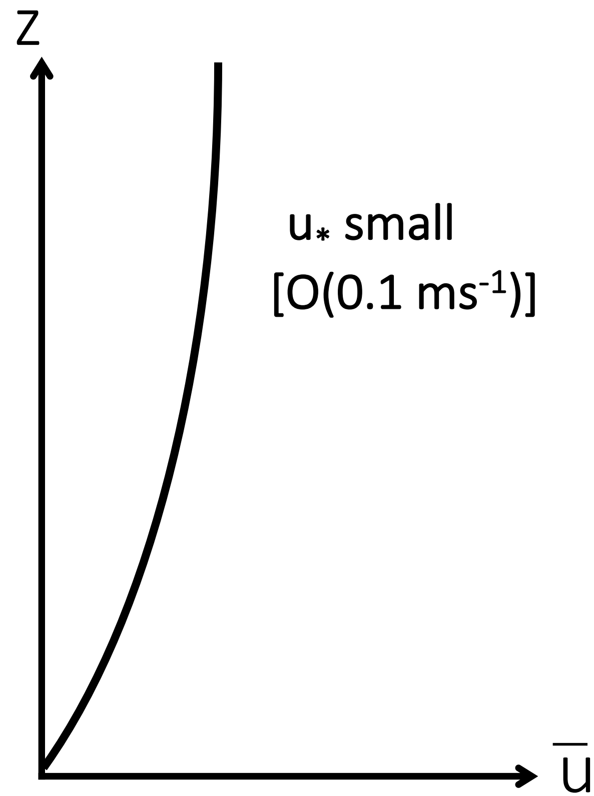
example: wind tunnel
→ surface A: hom roughness
→ lowest ca 10%:
‚constant stress‘

Raupach et al 1980

Friction velocity



strong wind
strong friction
(large gradients)



weak wind
weak friction
(small gradients)

Turbulence variables

- Reynolds decomposition and averaging:
  → co-variances = fluxes (turbulent transport)

- Intensity of turbulence

→ variances
→ standard deviations

- turbulence intensity $$I_k = \frac{\sigma_{u_k}}{\overline{u}}$$

Turbulence Kinetic Energy TKE

consider:

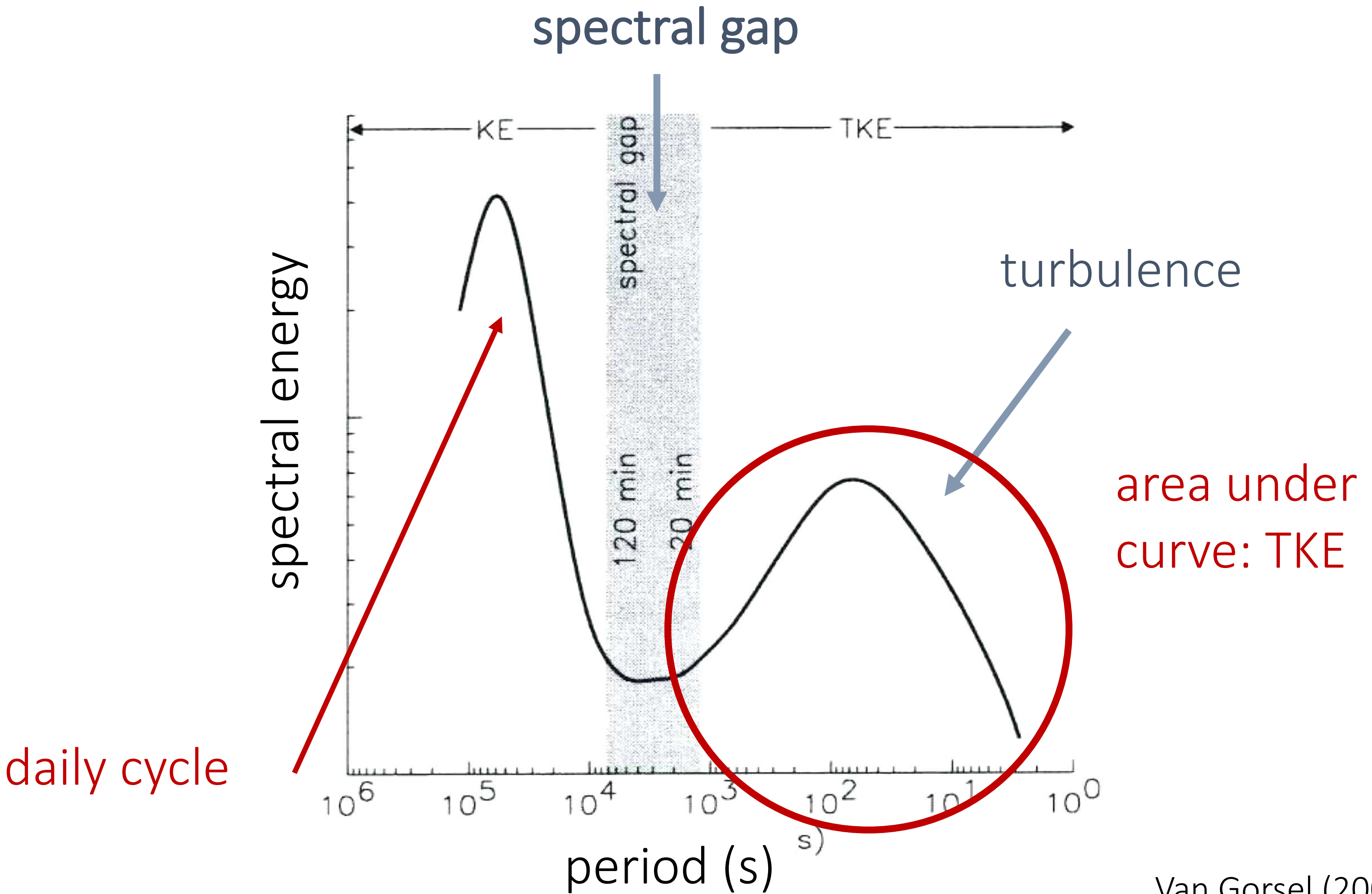
→ kinetic energy: $$E_{kin} = \frac{1}{2}\rho(u^2 + v^2 + w^2)$$

→ TKE: $$TKE = \frac{1}{2}\rho\,\overline{{u_{ij}}'^2}$$

→ how much kinetic energy in turbulence scales?

→ see spectra

Energy Spectra



Van Gorsel (2004)

Turbulence Potential Energy TPE

consider:

→ potential energy of mean flow:

$$E_{pot} = \int_0^\infty \rho g z \, dz = \mathrm{R}\int_0^\infty \rho T \, dz$$

→ TPE:

$$TPE = \frac{g}{\theta N^2}\frac{1}{2}\overline{\theta'^2}$$

→ mean potential temperature is proportional to T

→ turbulent potential temperature is proportional to q'^2

→ Zilitinkevich et al. 2007

Anisotropy of Reynolds stress tensor

→ go back to Reynolds stress tensor

$$\tau = \begin{bmatrix} uu & uv & uw \\ uv & vv & vw \\ uw & vw & ww \end{bmatrix}$$

→ symmetric tensor with 6 independent variables: fluxes and variances

→ Anisotropy: directional dependency

Anisotropy of Reynolds stress tensor

→ Isotropic: (i.e. invariant to rotation)

$$\tau = \begin{bmatrix} uu & 0 & 0 \\ 0 & vv & 0 \\ 0 & 0 & ww \end{bmatrix}$$

→ no off-diagonal terms (fluxes)

→ variances are the same $uu = vv = ww = \frac{2}{3} TKE$

→ Look only at the anisotropy stress (components)

$$b_{ij} = \frac{\overline{u_i' u_j'}}{\overline{u_l' u_l'}} - \frac{1}{3}\delta_{ij}$$

Anisotropy of Reynolds stress tensor

→ Tensor analysis: eigenvalues and eigenvectors (3)

→ **Anisotropy** can be described by a set of 2 **invariants** that are functions of eigenvalues of the anisotropy tensor (Lumley & Newmann 1977)

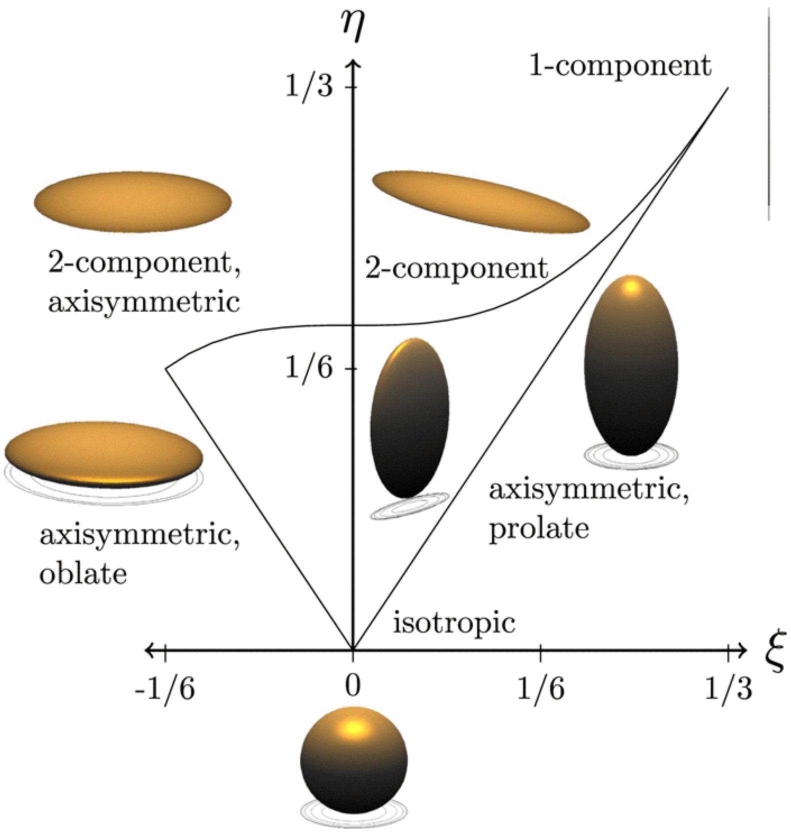
→ Invariants:

$$\eta^2 = \frac{1}{3}(\lambda_{\mathrm{I}}^2 + \lambda_{\mathrm{I}}\lambda_{\mathrm{II}} + \lambda_{\mathrm{II}}^2)$$

$$\xi^3 = -\frac{1}{2}\lambda_{\mathrm{I}}\lambda_{\mathrm{II}}(\lambda_{\mathrm{I}} + \lambda_{\mathrm{II}})$$

Anisotropy invariant maps

→ Lumley triangle



Hamilton and Cal (2015)

→ All realizable states of turbulence are within the map

→ 3 limiting states:

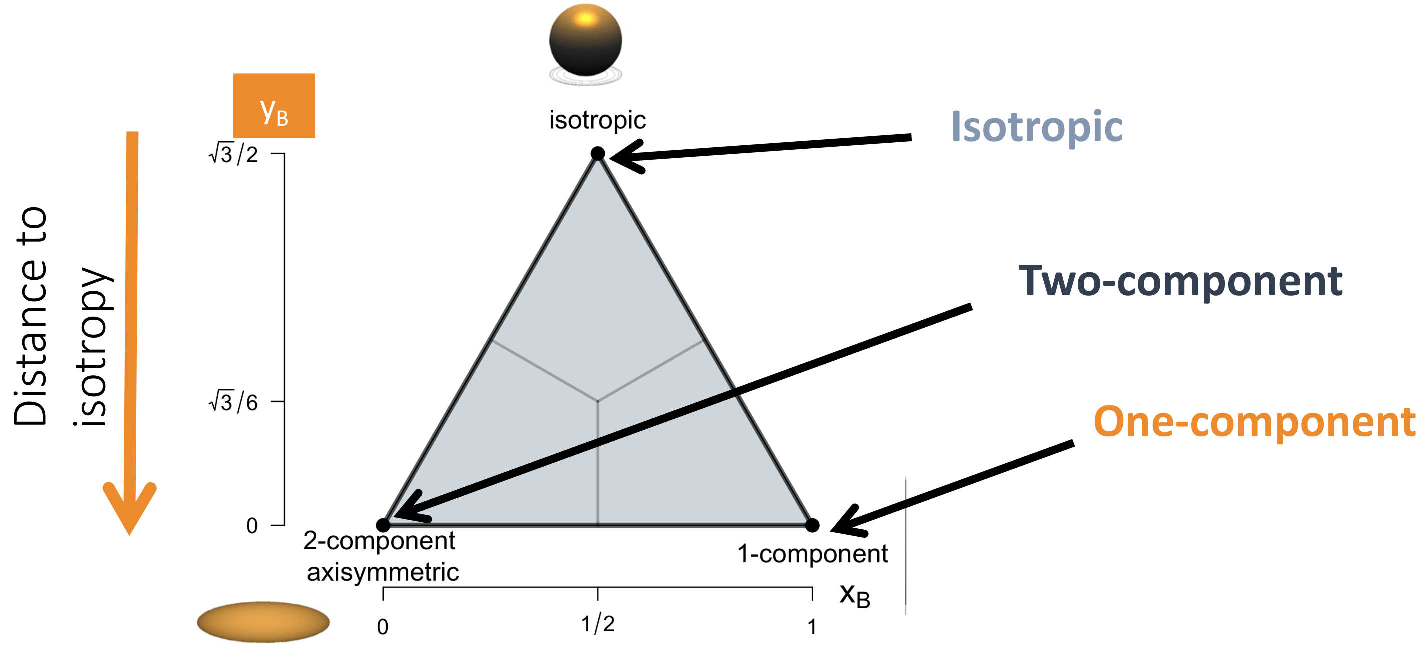
Isotropic

Two-component

One-component

Anisotropy invariant maps

→ Alternative representations: Barycentric map
→ Each limiting state occupies equal space



Banerjee et al. (2007)

Anisotropy of Reynolds stress tensor

Importance:

→ Anisotropy:

- Caused by forcing acting along different directions

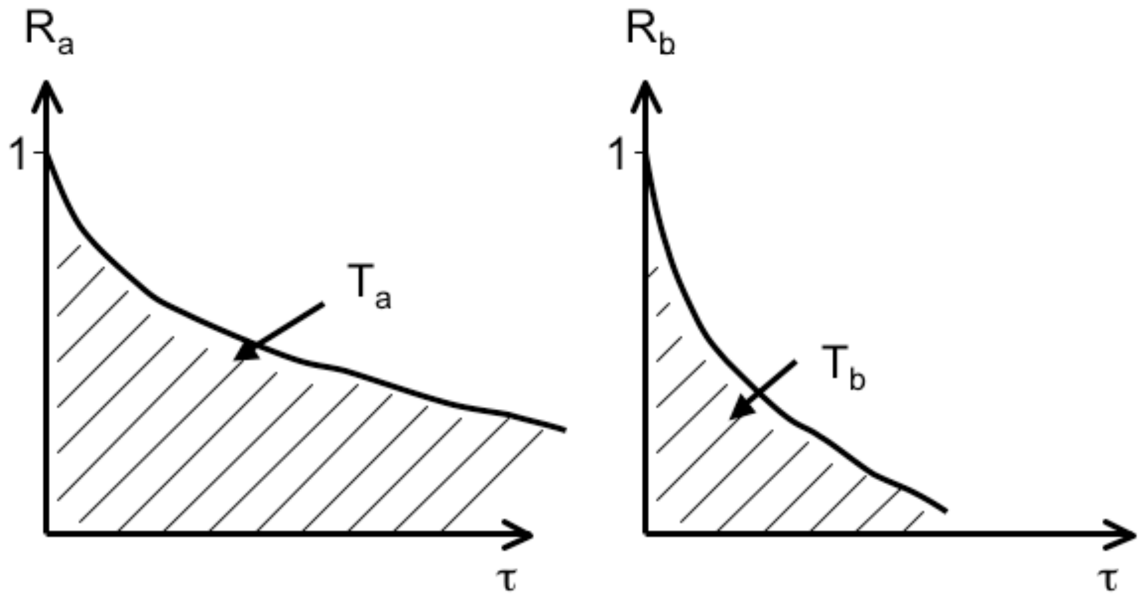
→ Isotropy:

- Inertial sub-range (chapter 7)
- Often assumed in models (equal contribution of all variances to TKE)

Turbulence variables

- auto-correlation function **Def:** $$R_a(t,\tau) = \frac{\overline{a'(t) \cdot a'(t+\tau)}}{\overline{a'^2}(t)}$$

→ t=0: $R_a(t,0) = 1$

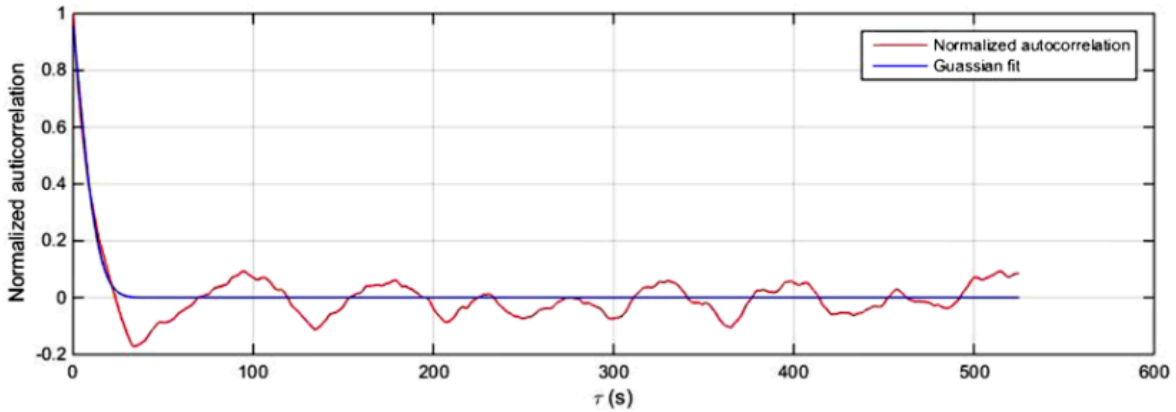


Integral time scale

- Def. $$T_a(\tau) =: \int_0^{\infty} R_a(\tau) d\tau$$

→ characteristic time
→ time scale over which turbulence remains correlated
'memory' of turbulence

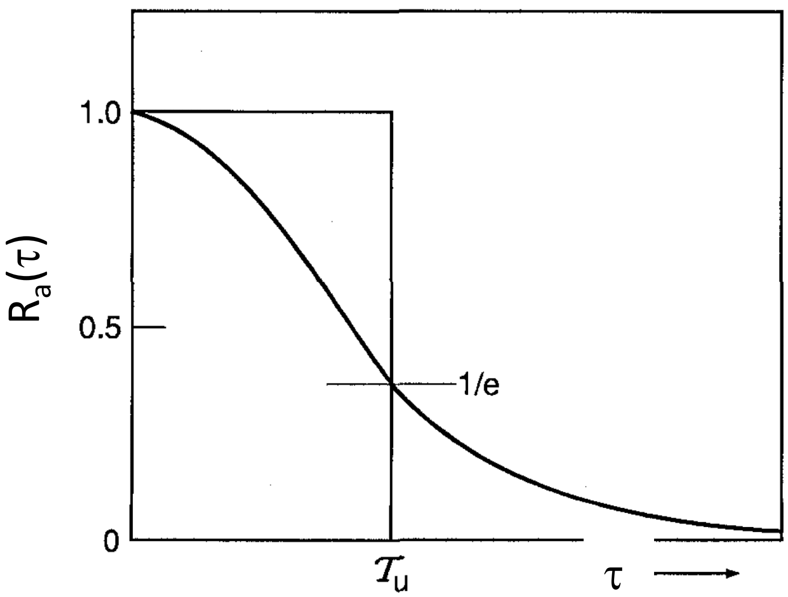
→ model (exponential decay) : $R_a(\tau) = \exp\{-\tau / T_a\}$



Tijera et al. 2017

Integral time scale

→ Alternative: find lag τ at which $R_a(\tau)$ = 1/e



Kaimal and Finnigan 1994

Integral length scale

Analogously:

$$R_{a,x}(x,\Delta x) =: \frac{\overline{a'(x)\cdot a'(x+\Delta x)}}{\overline{a'^2}(x)}$$

Integral length Scale:

$$L_{a,x}(\Delta x) =: \int_0^{\infty} R_{a,x}(\Delta x) d\Delta x$$

→ In practice (using Taylor's hypothesis)

$L_{a,x}(\Delta x) = T_a(\tau)\ U$

Summary: Statistical description

- cannot describe instantaneous fluctuations
  → statistical description, PDF
- Reynolds decomposition and averaging
  → co-variances:

$$\overline{(a \cdot b)} = \overline{(\overline{a} + a') \cdot (\overline{b} + b')}$$
$$= \overline{a} \cdot \overline{b} + \overline{a'b'}$$

  → meaning: turbulent transport
  → especially in the **vertical** (important)
- → sensible & latent heat, momentum, tracers
- turbulence kinetic and potential energy
- anisotropy
- integral time scales