6 Turbulent Kinetic Energy and Dynamical Stability

6.1 TKE-Equation

In the foregoing chapter an equation has been derived for the (summed) velocity variances that is repeated here for convenience in its flux form

$$\frac{\partial \overline{u_i'^2}}{\partial t} + \bar{u}_j \frac{\partial \overline{u_i'^2}}{\partial x_j} = -2\overline{u_i' u_j'} \frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \overline{u_j' u_i'^2}}{\partial x_j}$$

$$+2\delta_{i3}\overline{u_i'\left(\frac{\theta'}{\bar{\theta}}\right)}g + 2f_c\varepsilon_{ij3}\overline{u_i' u_j'} - \frac{2}{\bar{\rho}}\overline{u_i' \frac{\partial p'}{\partial x_i}} + 2\nu\overline{u'_i \frac{\partial^2 u'_i}{\partial x_j^2}} \qquad (5.34)$$

The last three terms on the rhs of (5.34) can be simplified as follows. Inserting the definition of ε_{ij3} (see definition in the note to Table 5.1) readily shows that the Coriolis term vanishes.

For the pressure term we note that

$$\frac{\partial (u'_i\, p')}{\partial x_i} = u'_i \frac{\partial p'}{\partial x_i} + p' \frac{\partial u'_i}{\partial x_i}, \qquad (6.1)$$

of which the last term on the rhs vanishes due to the summation (continuity equation). In flux form the second last term on the rhs of (5.34) therefore reads

$$-\frac{2}{\bar{\rho}}\overline{u'_i \frac{\partial p'}{\partial x_i}} = -\frac{2}{\bar{\rho}} \frac{\partial \overline{u'_i\, p'}}{\partial x_i}. \qquad (6.2)$$

For the last term on the rhs of (5.34) we note that

$$\frac{\partial^2 u_i'^2}{\partial x_j^2} = \frac{\partial}{\partial x_j}\left(2u'_i \frac{\partial u'_i}{\partial x_j}\right) = 2\left(\frac{\partial u'_i}{\partial x_j}\right)^2 + 2u'_i \frac{\partial^2 u'_i}{\partial x_j^2}. \qquad (6.3)$$

Therefore

$$2\nu u'_i \frac{\partial^2 u'_i}{\partial x_j^2} = \nu \frac{\partial^2 u_i'^2}{\partial x_j^2} - 2\nu\left(\frac{\partial u'_i}{\partial x_j}\right)^2, \qquad (6.4)$$

The first term on the rhs of (6.4) is on the order of 10^{-10} while the second term is $\mathcal{O}(10^{-3})$. The rate of dissipation of TKE is therefore defined as

$$\varepsilon =: \nu\overline{\left(\frac{\partial u'_i}{\partial x_j}\right)^2}, \qquad (6.5)$$

Recalling the definition of TKE (eq. 3.25), and with the above simplifications (5.34) yields a conservation equation for TKE per unit masss, e:

$$\frac{\partial e}{\partial t}+\overline{u}_j\frac{\partial e}{\partial x_j}=-\overline{u'_i u'_j}\frac{\partial \overline{u}_i}{\partial x_j}-\frac{\partial \overline{u'_j e}}{\partial x_j}+\delta_{i3}\overline{u'_i\theta'}\frac{g}{\overline{\overline{\theta}}}-\frac{1}{\overline{\rho}}\frac{\partial \overline{u'_j p'}}{\partial x_j}-\varepsilon \qquad (6.6)$$

The terms on the lhs (of (6.6) are the local temporal change and the advection term, respectively. On the rhs we have the *shear production*, the *turbulent transport of TKE*, the *buoyancy term*, the *pressure correlation term* and the *dissipation rate of TKE*.

Clearly, TKE is strongly dependent on the stability of the flow through the buoyancy term. Under convective (daytime) conditions the turbulent heat flux is positive $\overline{u'_3\theta'}>0$ and thus the buoyancy term is a production term. In contrary, under stable conditions (night time) this term becomes a sink and acts to damp TKE. Therefore, even under horizontally homogeneous conditions TKE exhibits a strong daily cycle (Fig. 6.1) and the local temporal change of TKE vanishes only during carefully selected periods of time.

Shear production of TKE

The first term on the rhs of (6.6) is always positive and thus a production term. This can easily be seen by recalling that – for example in the vertical direction – the shear stress is directed towards the surface ($\overline{u'_1 u'_3}<0$) and is the result of friction, which a the same time leads to a positive gradient of the mean wind speed ($\partial\overline{u}_1/\partial x_3>0$). Or in more general terms, in a Newtonian fluid the shear stress can be expressed as proportional to the rate of deformation. Therefore, all the individual products making up this term are negative and with minus sign we have a production term. Clearly, at least close to the surface the vertical shear stress makes up the dominant contribution to this term.

The shear production term describes the interaction of the turbulence with the mean flow. If the TKE equation has the form

$$\frac{\partial e}{\partial t}+\ldots=-\overline{u'_i u'_j}\frac{\partial \overline{u}_i}{\partial x_j}+\ldots.. \qquad (6.7)$$

the corresponding conservation equation for the *mean kinetic energy* ($E=0.5\overline{u}_i^2$) reads

$$\frac{\partial E}{\partial t}+\ldots=+\overline{u'_i u'_j}\frac{\partial \overline{u}_i}{\partial x_j}+\ldots.., \qquad (6.8)$$

thus showing that the shear production of TKE goes at the expense of mean kinetic energy of the flow.

Turbulent transport of TKE

First of all we note that – similar as with the second moments in the conservation equation for the mean variables – it is the *flux divergence* that enters the TKE equation. Assuming horizontally homogeneous conditions for the moment we further note that $\partial\overline{u'_3 e}/\partial x_3=\partial(\overline{u'_3 u'^2_1+u'_3 u'^2_2+u'^3_3}/\partial x_3$. Under stable/neutral conditions the third order moments and especially the skewness of the vertical velocity component are rather insignificant due to the near-normal distribution of the velocity fluctuations and correspondingly small is the turbulent transport term. In convective conditions, however, the non-local mixing through large eddies changes the situation. Figure 4.8 shows the profile of the skewness of the vertical velocity component under typical convective stratification. Figure 1.8 has already paved the terrain for the apprehension of large eddies being responsible relatively strong updrafts (thermals) over a limited area and weaker compensating subsidence, thus leading to a skewed distribution of the vertical velocity component.

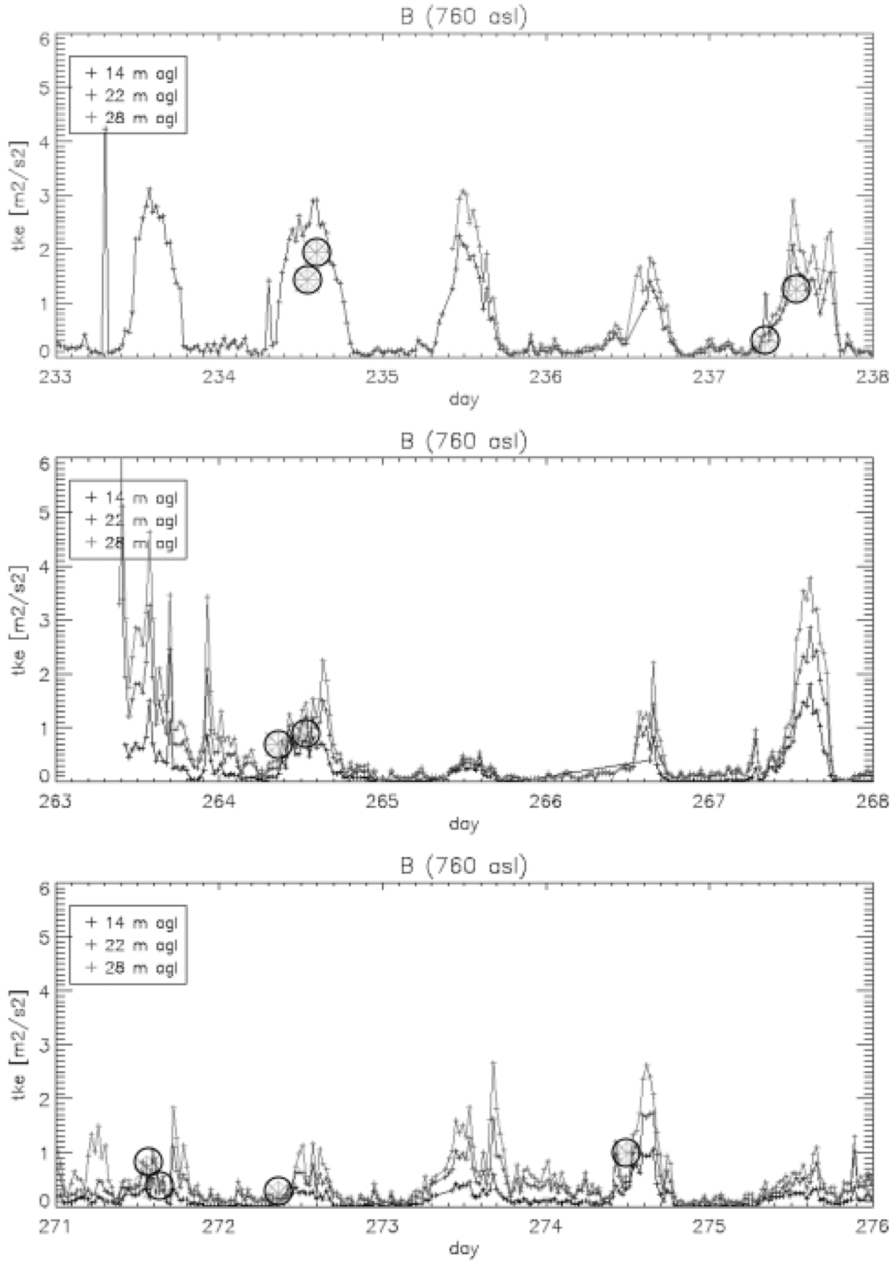

Figure 6.1 Time variation of TKE as measured at three different levels on tower (lines) and from a low flying light research aircraft (circled symbols). The numbers in the abscissa indicate the day-of-year in 1999 (from Weigel and Rotach 2004).

From Fig. 4.8 we see that close to the surface (essentially in the lower third or so of the CBL) $-\partial u_3'^3 / \partial x_3 < 0$ and hence turbulent transport of TKE constitutes a loss term. This is clear since the dominant (shear) production occurs close to the surface and through the transport term the upper parts of the boundary layer gain TKE. Figure 6.2 shows that also horizontal velocity variance is transported in a similar fashion away from the surface – and also from the top of the CBL downwards - through the transport term.

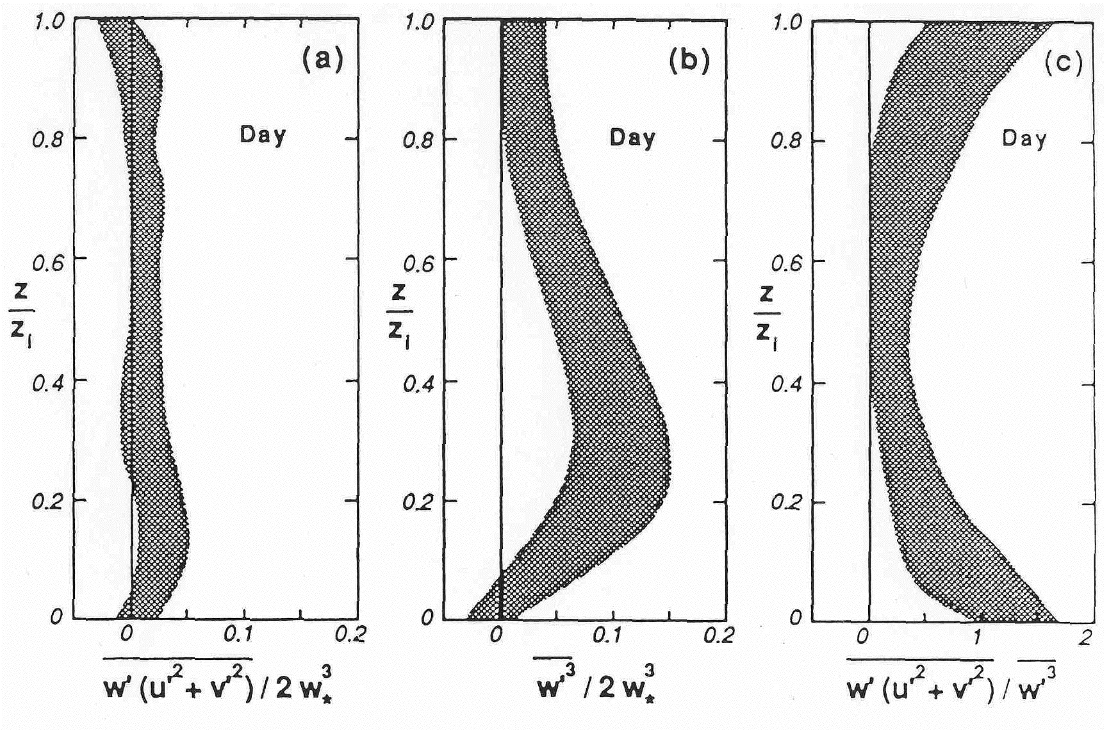

Figure 6.2 Vertical profiles of the mixed third moments in CBL's (shaded ranges). From Stull (1988).

The *pressure correlation term* is the least known of all contributions to the TKE budget, mainly due to observational problems. It is generally considered to be small and therefore often treated as a residual term in TKE budget studies.

Dissipation of TKE

The rate of dissipation of TKE is due to its definition (eq. 6.5) and meaning always negative. Close to the surface it is naturally at maximum (Fig. 6.3) due to the dominant shear production there. The dissipation of TKE eventually leads to kinetic energy of the flow being transformed into heat. Even close to the surface, where it is at maximum, this heat input is negligibly small as compared to all the other terms in the energy budget equation and therefore usually (and safely) neglected.

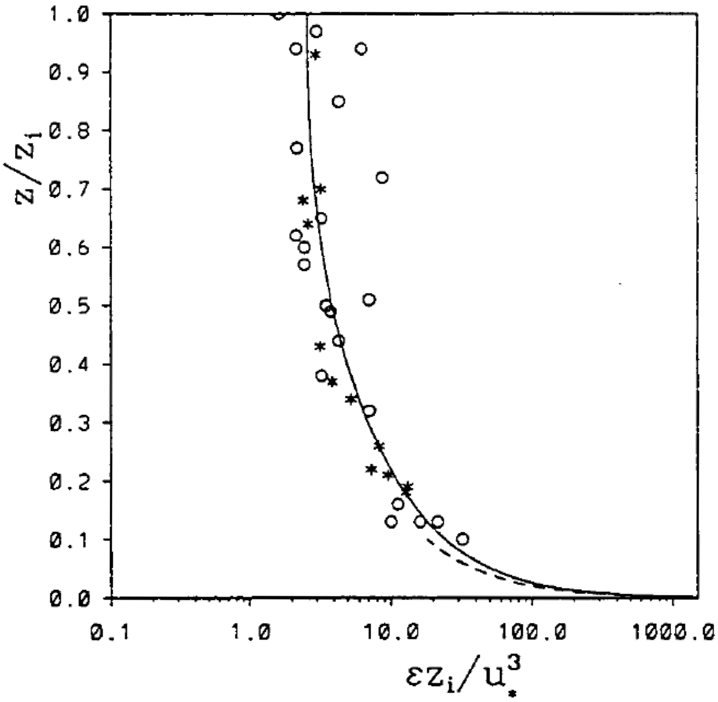

Figure 6.3 The near neutral profile for the dissipation rate of TKE, solid line, compared with measurements. Data: * Grant (1992), o Brost *et al* (1982). The dashed line represents a surface layer parameterization after Vogel and Frenzen (1992). From Rotach et al (1996).

Idealised profiles

Under ideally neutral, horizontal homogeneous and steady state conditions the shear production of TKE is balanced by dissipation (Fig. 6.4). Clearly, such conditions are difficult to meet in real flows and often other processes contribute to the TKE budget. The four dominating terms in near steady-state daytime ABL's are summarised in Fig. 6.5. Again it can be seen that close to the surface shear production and dissipation dominate and an important source of TKE in the central part of the boundary layer is turbulent transport away from the surface. The buoyancy term essentially follows the profile of the turbulent heat flux with a maximum at the surface and a minimum (loss of TKE due to the entrainment process) at the top. After sunset the buoyancy term also becomes a sink (Fig. 6.6, left panel) and shear production is way too weak to maintain the TKE levels of the day. Therefore, after some hours TKE levels are largely reduced and so are the budget terms (Fig. 6.6, right panel).

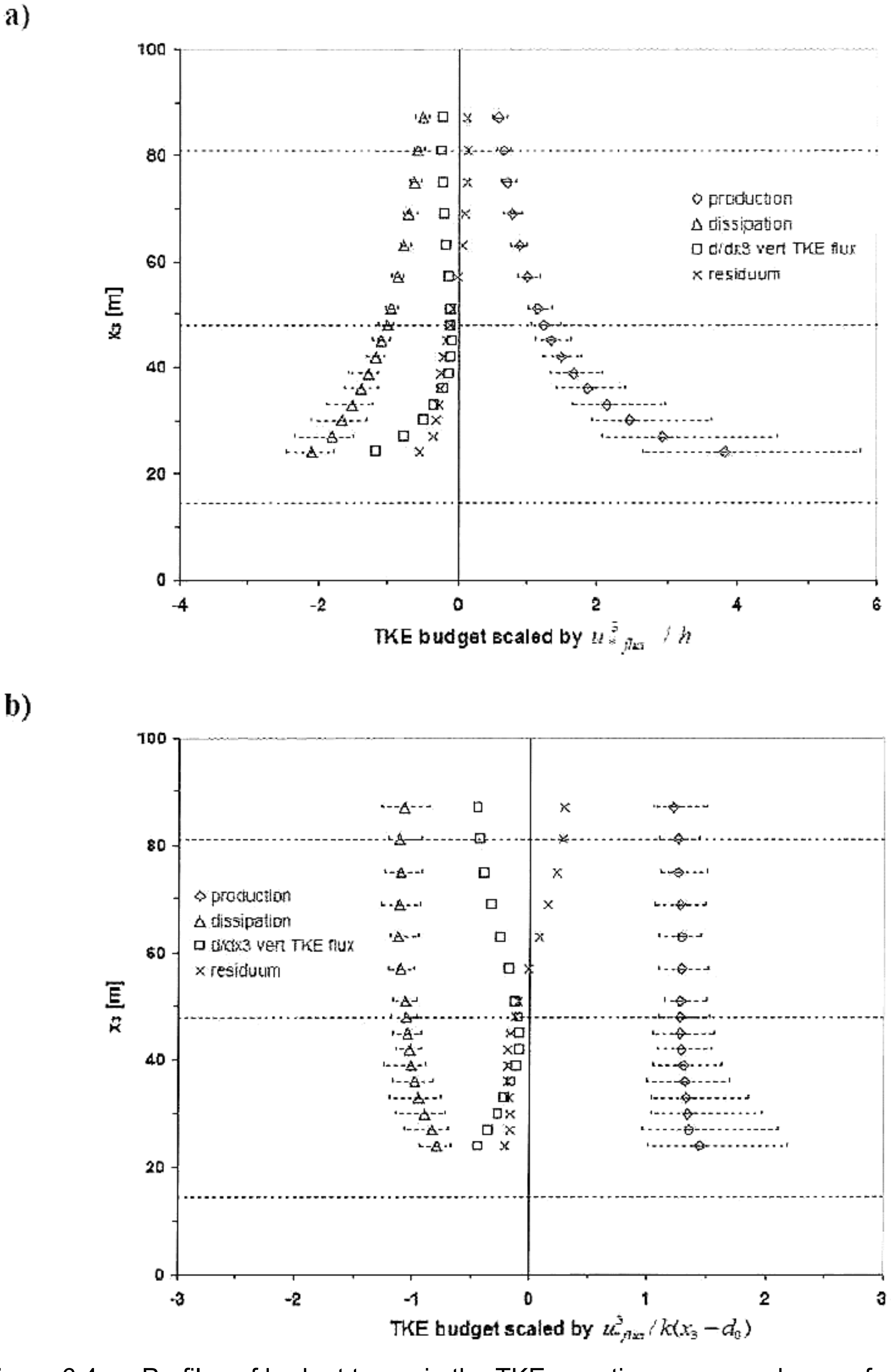

Figure 6.4 Profiles of budget terms in the TKE equation over an urban surface in a wind tunnel. The dashed lines indicate (from the bottom upwards) the mean building height, the height of the roughness sublayer (section 8.2) and the height of the inertial sublayer. From Feddersen (2005).

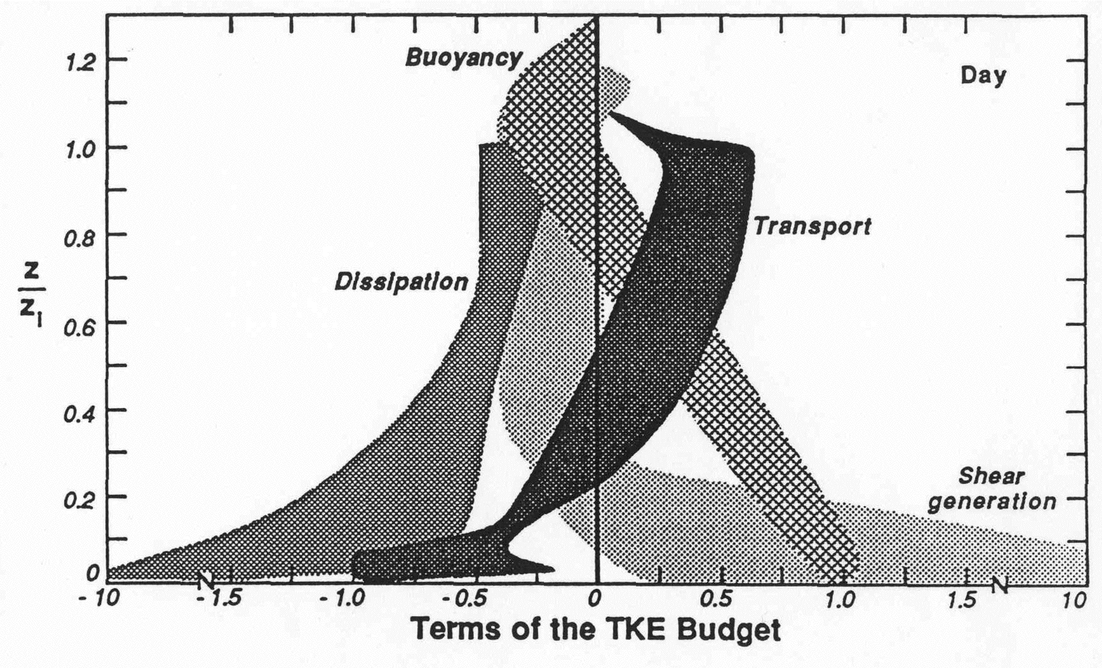

Figure 6.5 Vertical profiles of budget terms in the TKE equation under convective conditions (shaded areas give spread). From Stull (1988).

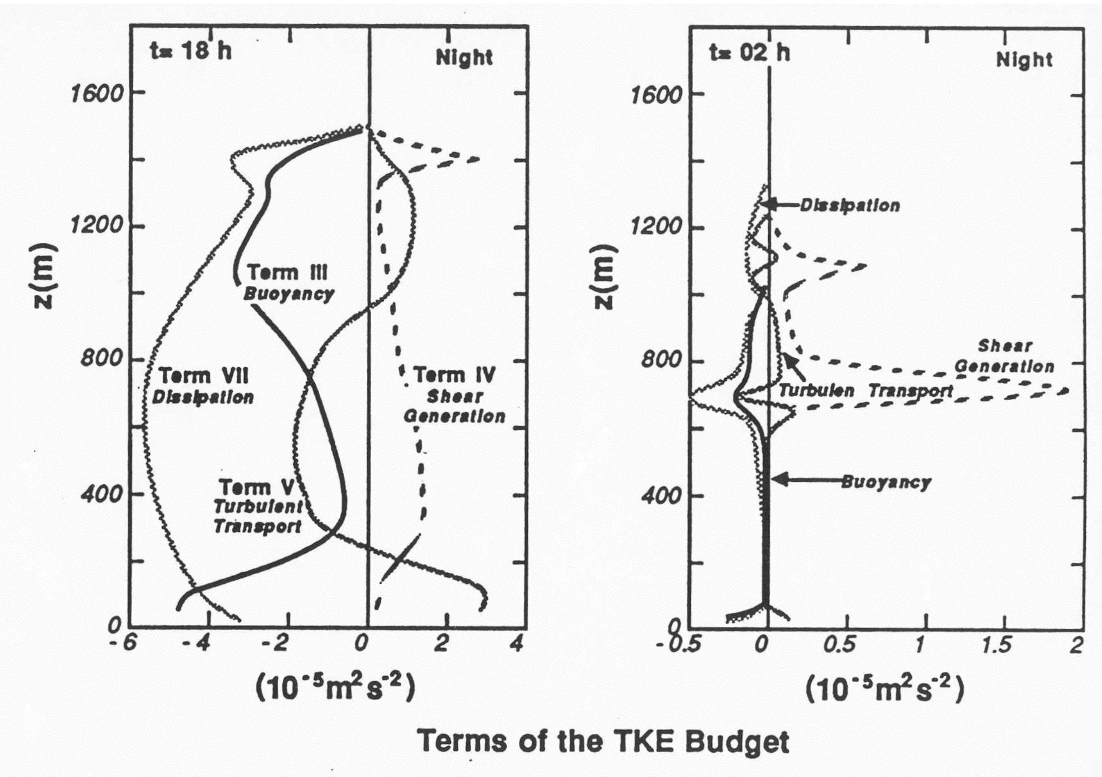

Figure 6.6 Modelled budget terms of the TKE equation during 'night 33-34' in the Wangara experiment. Left panel at 6 pm (day 33) and right panel at 2 am (day 34). From Stull (1988).

6.2 Stability Measures

Until now we have used stability as a discriminating criterion for different types of boundary layer states. In this we have always silently assumed that with 'stability' the *static stability*, i.e. the gradient of the potential temperature was referred to. However, the stability of a turbulent flow not only depends on the thermal stratification but also on the shear production of TKE, which is even dominating in various regions of the ABL. In loos terms we may say that a turbulent flow is very unstable if shear production is supported by additional buoyancy production, is near-neutral if the buoyancy term is small and finally is stable if buoyancy acts to damp TKE. It seems therefore appropriate to define a *dynamical stability measure*, which takes into account not only the sign of the buoyancy production/damping term but also the strength of the shear production.

6.2.1 The Flux Richardson number

One of the pioneer's of atmospheric turbulence research, L.F. Richardson, has been the first to introduce a stability measure based on the TKE budget equation (6.6). His simplifying (idealising) assumptions were to consider a quasi-stationary ($\partial/\partial t = 0$), horizontally homogeneous ($\partial/\partial x_1 = \partial/\partial x_2 = 0$) flow without subsidence ($\overline{u}_3 = 0$) and a coordinate system aligned with the mean wind ($\overline{u}_2 = 0$). With this eq. (6.6) becomes

$$0 = -\overline{u'_1 u'_3}\frac{\partial \overline{u}_1}{\partial x_3} - \frac{\partial \overline{u'_3 e}}{\partial x_3} + \overline{u'_3 \theta'}\frac{g}{\overline{\overline{\theta}}} - \frac{1}{\overline{\rho}}\frac{\partial \overline{u'_3 p'}}{\partial x_3} - \varepsilon . \tag{6.9}$$

Based on the this idealized TKE budget equation Richardson identified the shear production and the buoyancy terms as the dominating terms, based on which he defined a dynamical stability measure that is since then known as the *Flux Richardson Number*, R_f:

$$R_f =: \frac{g}{\overline{\overline{\theta}}}\frac{\overline{u'_3 \theta'}}{\overline{u'_1 u'_3}\frac{\partial \overline{u}_1}{\partial x_3}} . \tag{6.10}$$

Even if the assumption of neglecting especially the dissipation term may not prove particularly good (cf. Fig. 5.4) R_f has become an essential variable in defining the turbulence state of atmospheric flows. Its most obvious property is that $R_f < 0$ for *unstable* conditions, $R_f = 0$ for *neutral* flows and $R_f > 0$ in stable conditions. If both shear production and buoyancy contribute to TKE production there is no theoretical limitation to R_f (at least not in the idealised assumption of its derivation) although it is hardly observed to become smaller than –10. On the stable side we readily observe that production of TKE is larger than damping if $R_f < +1$. For larger Richardson numbers, no turbulence can be maintained even if shear production should exist, simply because it is readily damped away by the stratification. For $0 < R_f < 1$ the flow is statically stable ($\partial\overline{\theta}/\partial x_3 > 0$) and dynamically unstable in the sense that turbulence can exist. Still, this consideration has neglected the dissipation of TKE, which certainly (and substantially) will contribute to the suppression of turbulence. Equating the total production (still only shear production) with the 'total' suppression (buoyancy plus dissipation terms) yields an expression for a 'critical' state, in which turbulence can 'just' be maintained:

$$\frac{\frac{g}{\overline{\theta}}\overline{u'_3\,\theta'}-\varepsilon}{\overline{u'_1\,u'_3}\,\frac{\partial \overline{u}_1}{\partial x_3}}=1. \tag{6.11}$$

Equation (6.11) can be expressed in terms of R_f to yield a 'critical' Richardson number above which damping of TKE is dominating over production

$$R_{f,crit}=1+\frac{\varepsilon}{\overline{u'_1\,u'_3}\,\frac{\partial \overline{u}_1}{\partial x_3}}. \tag{6.12}$$

The sign of the second term on the rhs of (6.12) is always negative thus indicating that somewhere between $0<R_f<1$ this critical state is reached (see below, 6.2.2). Beyond this point, turbulence occurs only sporadically, i.e. it may be produced locally and is 'dissipated away' quite quickly. In terms of scaling regimes (Fig. 4.4) this state of the stable boundary layer is referred to as *intermittency*.

6.2.2 The Gradient Richardson number

The diagnostics of the dynamic stability using the Flux Richardson number requires the knowledge of the turbulent fluxes (eq. 6.10), which are often not available (e.g., in a numerical model with a first order turbulence closure). Also, as a measure for the transition between laminar and turbulent states of a flow, a measure might be desirable that is non-zero in both[1]. Therefore as an approximation, K-Theory (Section 5.2.1) is used to define the so-called *Gradient Richardson Number*, Ri. Thereby the simplifying assumption is made that the exchange coefficients for momentum, K_m, and for sensible heat K_H are equal, thus

$$Ri=:\frac{g}{\overline{\theta}}\frac{\partial\overline{\theta}/\partial x_3}{(\partial \overline{u}_1/\partial x_3)^2}. \tag{6.13}$$

Clearly, Ri is easier to determine than R_f but it still has the 'theoretical' foundation of employing in an idealised fashion the TKE budget equation for its definition. It has the same properties in terms of sign as the Flux Richardson number (positive for statically stable and negative for statically unstable conditions). The flux and Gradient Richardson numbers are related through

[1] Note that the turbulent fluxes are zero in laminar flows and hence is the Flux Richardson Number

$$R_f = Ri\,\frac{K_H}{K_m}\,. \qquad (6.14)$$

In the literature a critical Richardson number (cf. 6.12) is always given in terms of Ri rather than R_f. Observational evidence and some further theoretical considerations (Nieuwstadt, 1984) indicate that

$$Ri_c \approx 0.25, \qquad (6.15)$$

i.e. at a value substantially smaller than one the action of buoyancy and dissipation make it impossible to maintain a fully turbulent state.

6.2.3 The Bulk Richardson number

If the evaluation of gradients is still beyond the possibilities an even simpler approach to dynamic stability is the *Bulk Richardson Number*, Ri_B. In this further simplification the gradients are replaced by differences, and hence

$$Ri_B =: \frac{g}{\bar{\theta}}\frac{\Delta\bar{\theta}\Delta x_3}{(\Delta\bar{u}_1)^2}. \qquad (6.16)$$

The Bulk Richardson number is often used as an approximation for the entire boundary layer and hence the differences are taken between the ABL top and the surface (in this case $\Delta\bar{u}_1$ reduces to $\bar{u}_1$ because the mean speed vanishes at the surface). Clearly Ri_B has the advantage of simplicity but the linearization implied in the transition between Ri and Ri_B is not generally based on solid ground.

6.2.4 Stability measure in the Surface Layer

In Chapter 4 the non-dimensional quantity z/L, where L is the Obukhov length, has been found to be the 'one and only' dimensionless group to describe turbulence variables in the *Surface Layer*. This result was achieved using similarity theory. We may use this result to expressing the simplified TKE budget (6.9) in terms of SL variables and in non-dimensional form. For this we multiply each term by kx_3/u_*^3 and replace the turbulent fluxes by their surface values ($-\overline{u'_1u'_3} \to -(\overline{u'_1u'_3})_o = u_*^2$ and $-\overline{u'_3\theta'} \to -(\overline{u'_3\theta'})_o$:

$$\begin{aligned} 0 &= \frac{kx_3}{u_*}\frac{\partial\bar{u}_1}{\partial x_3} - \frac{kx_3}{u_*^3}\frac{\partial\overline{u'_3e}}{\partial x_3} + \frac{kx_3 g(\overline{u'_3\theta'})_o}{\bar{\theta}u_*^3} - \frac{kx_3}{\bar{\rho}u_*^3}\frac{\partial\overline{u'_3p'}}{\partial x_3} - \frac{kx_3}{u_*^3}\varepsilon \\ &= \Phi_m - \Phi_{tr} - \frac{z}{L} - \Phi_p - \Phi_\varepsilon \end{aligned} \qquad (6.17)$$

The chosen scaling exactly[2] corresponds to Richardson's approach in that the former buoyancy term now corresponds to z/L and expresses the ratio between buoyancy production/damping and shear production of TKE.

Monin-Obukhov similarity theory then predicts that all the 'Phi functions' in (6.17) be a function of z/L alone – what has been shown to be a good

[2] With the additional von Kàrmàn constant and the constraint of surface fluxes.

prediction for, e.g. Φ_m (Fig. 4.3) and a useful approximation for Φ_ε even outside the SL (Fig. 6.3) under near-neutral stratification. Figure 6.7 shows the dominating terms (with the transport and pressure terms as residual) over a horizontally homogeneous snow surface on the Greenland ice sheet. Neglecting the turbulent transport and pressure correlation terms for a moment we find from this analysis that the sum 'shear production minus dissipation' in its non-dimensional form is linearly related to z/L, an observation that is often confirmed to approximately hold in the SL.

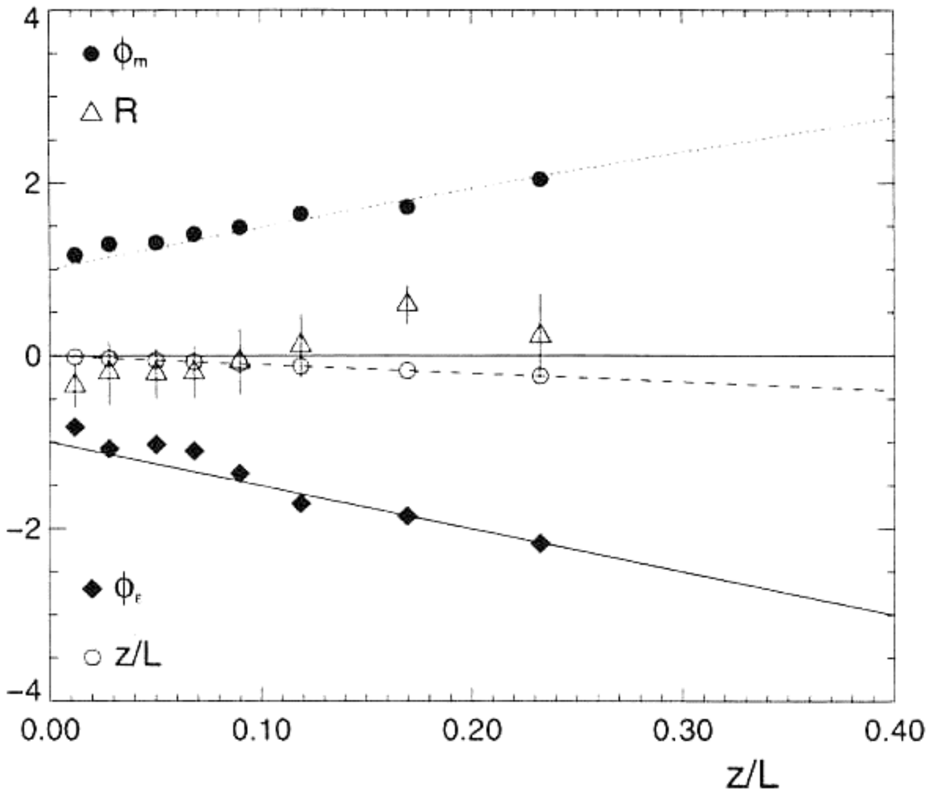

Fig. 6.7 Terms of the non-dimensional TKE budget equation in the SL over a horizontally homogeneous snow-covered surface on the Greenland ice sheet. The dotted line corresponds to (4.24), the solid line is a parameterization (stable stratification) for the non-dimensional dissipation rate, the dashed line is the 1:1 line for the buoyancy term. The triangles denote the residuum term. Symbols as bins over stability ranges. From Forrer (1999).

Finally, using the non-dimensional wind shear (eq. 4.14) and the non-dimensional temperature gradient (eq. 4.25) we can find a relation between Ri and z/L:

$$Ri = \frac{z}{L}\frac{\Phi_H}{\Phi_m^2}. \qquad (6.18)$$

And with (6.14) we obtain

$$R_f = \frac{K_H}{K_m} Ri = \frac{x_3}{L} \frac{\Phi_H}{(\Phi_m)^2} \frac{K_H}{K_m}. \qquad (6.19)$$

These equations can be used if, e.g. gradients of wind speed and temperature are available from which z/L is derived iteratively.

References Chapter 6

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