3 Statistical Treatment of Turbulence

3.1 Averaging, Stationarity and Homogeneity

Due to the chaotic nature of turbulent flows it is impractical to try to describe in detail every single trajectory in a turbulent flow. Even if this restriction is a matter of CPU time and one particular realisation of a turbulent flow field can in principle be obtained from numerical simulation of sufficiently high spatial and temporal resolution, alternative approaches must be sought. The most powerful of these is to consider turbulence variables as a realisation of a stochastic process. Thus, rather than looking at the variable's value, its probability density function (pdf), P, which contains information on the statistical distribution of occurrences of particular values, is considered. Figure 3.1 shows what a measurement of a variable at sufficiently high frequency yields if placed in a turbulent flow. Clearly, it can be seen that measurement values change at very high rate and that both the mean value and the variance change slowly in time with respect to the time scale of turbulence. If P_a is the probability that variable a assumes a certain value, the average can be determined according to

$$\overline{a} = \int_{-\infty}^{\infty} a P_a da. \tag{3.1}$$

Here, the overbar is introduced to refer to an average. In other words, each possible value is considered with its probability of occurrence and the integration goes over the entire phase space. In general, we may estimate the average of any function of variable x according to

$$\overline{f(a)} = \int_{-\infty}^{\infty} f(a) P_a da. \tag{3.2}$$

A pdf is defined through all its n moments:

$$\overline{a^n} = \int_{-\infty}^{\infty} a^n P_a da, \tag{3.3}$$

where n=0 corresponds to the normalisation of the pdf, n=1 to the average and so on. Useful measures are further the *central moments*, defined as

$$\overline{(a-\overline{a})^n} = \int_{-\infty}^{\infty} (a-\overline{a})^n P_a dx, \tag{3.4}$$

with n=0 again corresponding to the norm. n=1 corresponds to the average of the fluctuations around the mean and is thus zero by definition. The *variance* is obtained for n=2 and the *skewness* for n=3, which becomes zero for a symmetric distribution. Finally, n=4 corresponds to the so-called *flatness* of the pdf.

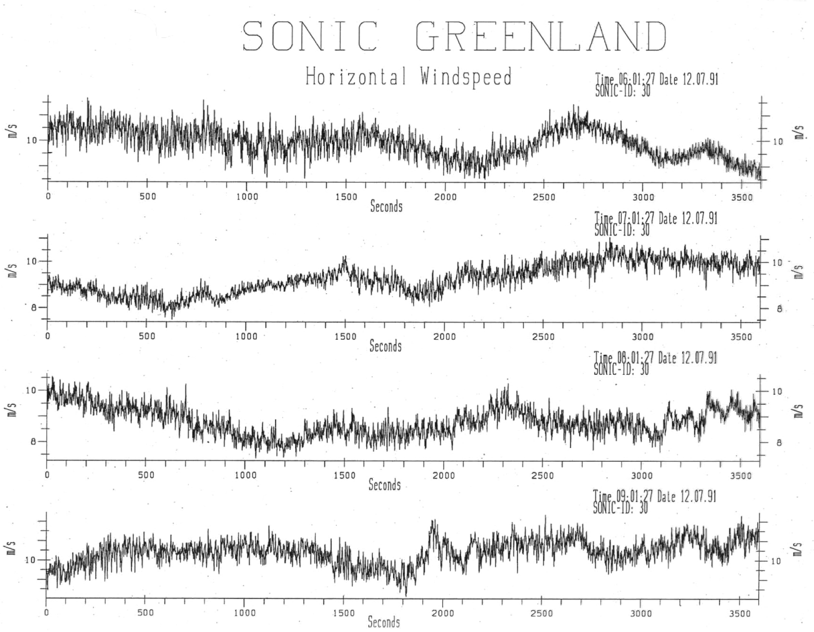

Figure 3.1 Time series of wind speed component in a turbulent flow (from a measurement at 'ETH camp' on the Greenland ice sheet). Each row corresponds to 1 hr worth of data.

The best-known pdf is certainly a Gaussian or normal distribution, for which the skewness is zero and the flatness equals 3 (all other higher moments vanish). Many turbulence variables are indeed close to normally distributed – and even more often assumed to be so. Still, some important processes in the ABL lead to non-Gaussian distributions. For example, in a CBL usually strong updrafts (thermals) are present but they occupy only a relatively small portion of the surface. Over the remainder of the area compensating negative vertical wind is prevailing. Thus, the most likely value (i.e., the median) observed for w is slightly negative and large negative values are unlikely. Large positive values, on the other hand, occur more often (in the thermals) and the pdf of w is positively skewed (Fig. 3.2).

3.1.1 Ensemble averages

What we really want when measuring or modelling a turbulence variable is to obtain information on the processes, which lead to *such a time series* – but not exactly the one we obtained. We could have chosen our site a few centimetres to one side or we could have started the observation a few seconds later. Therefore, the average we are truly interested in is that we would obtain if we had the possibility to repeat our measurement (or simulation) over *all possible realisations* of the flow. This is called an *ensemble average*. Formally, we may write for the ensemble average of variable a

$$\overline{a}^e = \frac{1}{N}\sum_{i=1}^{N} a_i(\vec{x},t). \tag{3.5}$$

Here, a_i denotes an individual realization of variable a as a function of space and time, and N is the number of realization. Unfortunately, in the atmosphere, as in any environmental system, N goes to infinity and thus a true ensemble average can hardly be obtained. The so-called *Ergodic hypothesis* states that under certain conditions (see Panofsky and Dutton 1984, p. 61ff for details) a spatial average, $\overline{a}^x$ or a temporal average $\overline{a}^t$ can be taken as a surrogate for the ensemble average. It is beyond the scope of this book to detail all these conditions, but one must be mentioned: stationarity. Implicitly, the Ergodic hypothesis is always invoked when measuring or modelling[1] turbulence variables in order to study processes in turbulent flows.

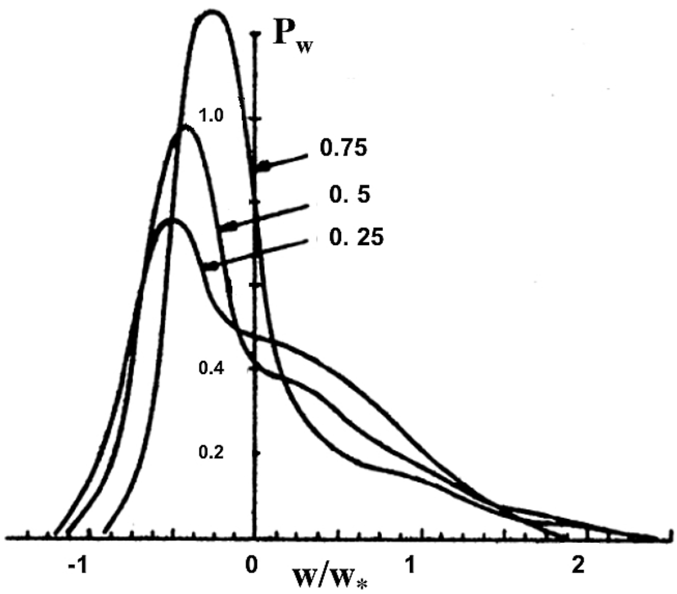

Figure 3.2 Skewed pdf of the vertical velocity in a CBL. Different lines for different non-dimensional heights (see labels). From Lamb (1982).

Over a sufficiently homogeneous surface, *S*, a spatial average may be defined according to

$$\overline{a}^x = \frac{1}{S}\iint_S a(\vec{x},t)ds. \quad (3.6)$$

Similarly, a temporal average over a stationary period *T* can be obtained from

$$\overline{a}^t = \frac{1}{T}\int_{t_1}^{t_1+T} a(\vec{x},t)dt. \quad (3.7)$$

The spatial average in the ensemble sense is what is usually desired in order to investigate processes leading to the particular structure of turbulence in the environment under consideration. However, not only due to the requirement of spatial homogeneity this is usually difficult to obtain. Also, a large number of

[1] No Ergodic hypothesis is necessary, of course, if D*irect Numerical Simulation* (DNS) is employed, i.e. if a flow is simulated resolving all scales (down to the smallest) of turbulence. In such a case, the model simulation yields *one possible realisation* of the flow and may, in principle be repeated manifold. The same is true, to a certain degree, for *Large Eddy Simulation*, in which scales down to a certain limiting size are resolved.

instruments in the field are necessary. In recent time, with turbulence instrumentation becoming more easily available, highly sophisticated spatial arrangements of instruments are employed (e.g., Horst et al. 2004), often in connection with testing very sophisticated numerical models. Still, what we can more easily obtain is a temporal average from a single instrument. Thus very often a single instrument reading is employed to get $\overline{a}^t$ and the result is interpreted as the desired ensemble average $\overline{a}^e$ or simply $\overline{a}$. Some useful rules for averaging are summarised in Table 3.1.

Table 3.1: Useful rules for averaging

A, B are variables, *c* is a constant	
$\overline{c} = c$	
$\overline{(c \cdot A)} = c \cdot \overline{A}$	
$\overline{(\overline{A})} = \overline{A}$	An average behaves like a constant
$\overline{(\overline{A} \cdot B)} = \overline{A} \cdot \overline{B}$	
$\overline{(A \cdot B)} \neq \overline{A} \cdot \overline{B}$	The average of a product is not, in general, the product of the averages
$\overline{(A + B)} = \overline{A} + \overline{B}$	
$\overline{\left(\frac{\partial A}{\partial x}\right)} = \frac{\partial \overline{A}}{\partial x}$	This is an important property and derives from the Leibnitz theorem.

For the above concepts, *stationarity* has been assumed as a prerequisite. How can this be defined? Consider the average temporal correlation of a fluctuating variable a[2] with itself

$$\overline{a(t) \cdot a(t')} =: C_a(\tau, t), \quad (3.8)$$

where C_a is called the auto-covariance function, $\tau = t - t'$ and t is the absolute time. For stationary turbulence C_a must not depend on t, i.e.

$$\overline{a(t) \cdot a(t')} =: C_a(\tau), \quad (3.9)$$

and this must be true for *any time difference* τ. Thus it must also be valid for $\tau = 0$ and in this case (3.9) describes nothing else than the variance. It follows that if the variance is not dependent on t, this is a condition for stationarity. However, is this sufficient to define stationarity? We have stated above that the pdf of a turbulence variable is defined through all its moments. Hence in principle we have to repeat the above procedure for all moments to find: a

[2] That is, for the time being, the instantaneous value of A of which the mean value $\overline{A}$ is subtracted. Later (in Section 3.3) we will introduce this procedure as Reynolds decomposition.

time series is stationary, if all its moments do not depend on t. Clearly, in practice only the first few (often up to the second) are tested in order to ensure stationarity.

However, in Chapter 2 we have identified the dissipative nature of turbulence as one of the ingredients of the 'turbulence syndrome'. Thus turbulence is never truly stationary due to the continuous production and dissipation of turbulent kinetic energy. A practical solution is therefore to introduce the concept of *quasi-stationarity*. For this we consider two time scales (Nieuwstadt and Duynkerke 1996). The first, T_f, is the forcing time scale, which describes the external processes driving the boundary layer flow. Often, it may be assumed that

$$T_f \propto f^{-1}, \qquad (3.10)$$

where f is the Coriolis parameter, thus reflecting the fact that some external forcing on the order of the Geostrophic wind is present. In mid-latitudes f is roughly 10^{-4} s^{-1} and T_f becomes some 3 hours. A second time scale is T_m reflecting the time it takes to change a mean profile due to a change in the boundary conditions. Now, if

$$T_m << T_f \qquad (3.11)$$

the process is called quasi-stationary. For example for the surface layer, a characteristic velocity u_* has been introduced and later we will learn that u_* is on the order of a few tenth of ms^{-1}. If the height z is considered as a length scale[3], T_m can be estimated for the surface layer using simple dimensional arguments, $T_m = z/u_*$. And it becomes a few tens or hundreds of seconds. Thus, according to (3.11) turbulence in the SL *may* be quasi-stationary, even if this is no guarantee that it really is under any circumstances.

Considering spatial *homogeneity* of a turbulent flow the same arguments as for stationarity can be invoked – simply considering the spatial auto-covariance function instead of (3.8) and, in principle, investigating it up to all the moments of the pdf. Thus, homogeneity is nothing else than stationarity in space. This is also obvious when considering surfaces of different types (Fig. 3.3). Over a homogeneous surface, a stationary flow may establish even close to the surface (i.e., so close to the roughness elements that the flow is dynamically influenced by them) while the continuously changing surface forcing over an truly inhomogeneous surface (random and largely different roughness elements) will not allow the stationarity conditions to be fulfilled. Clearly, the mean advection velocity (mean wind speed) plays a role herein as it determines the time scales the flow needs to travel from one surface type to another. At a certain distance from the surface the impact of different roughness elements will have dispersed through turbulent mixing into the overall-response of this type of surface. This distance is often referred to as *blending height*.

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[3] This choice will be substantiated in Chapter 4.

It is important to note here that homogeneity is a concept in boundary layer meteorology that is only invoked for the horizontal dimensions. By 'definition' the vertical dimension is expected to be inhomogeneous due to the presence of the Earth's surface and hence the existence of the boundary layer with its very characteristics.

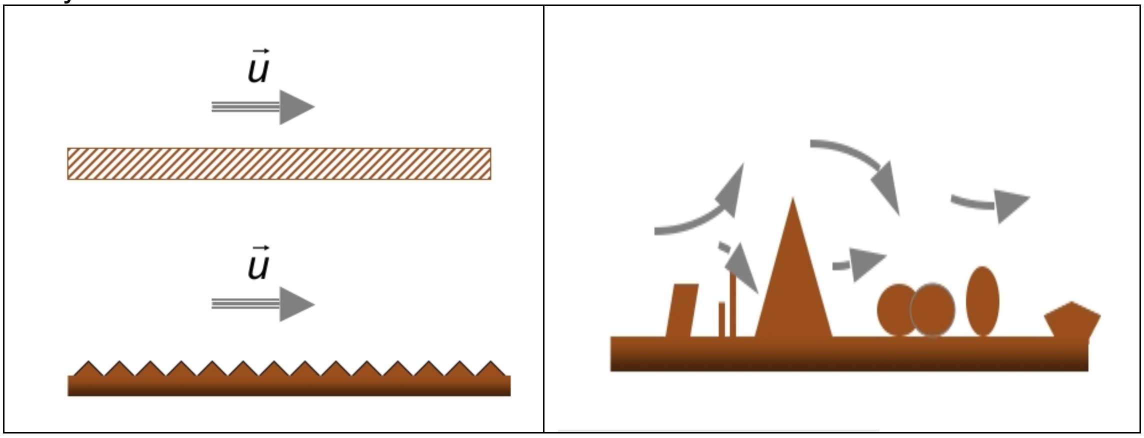

Figure 3.3 Over a regular (horizontally homogeneous) surface a stationary flow can establish (left), while the flow has to readjust on each of the varying roughness elements and will thus not become stationary (right).

3.2 Taylor Hypothesis

Considering the various types of averages above (ensemble vs. time vs. space) we often have the situation that we are, in fact, interested in the (spatial) structure of turbulence but we have at hand a time series from possibly only one instrument (or, alternatively a modelled time series). In other words the problem is how to 'observe' the spatial structure without truly spatially resolved information. In 1838 Geoffrey I. Taylor formulated his famous *hypothesis* as a way out of this dilemma: If the turbulence can be assumed to be '*frozen*' during the time it travels across the point of observation, the temporal information can easily be converted into spatial information. Figure 3.4 illustrates this concept by considering an idealised eddy as it passes a hypothetical sensor.

Mathematically, Taylor's hypothesis is most conveniently expressed using the total (Lagrangian) time derivative for a variable ζ:

$$D\zeta / Dt = 0 \tag{3.12}$$

Expanding this equation into its components

$$\frac{D\zeta}{Dt} = \frac{\partial \zeta}{\partial t} + \frac{\partial \zeta}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \zeta}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \zeta}{\partial z}\frac{\partial z}{\partial t} = 0 \tag{3.13}$$

and identifying the time derivatives of the coordinate axes as the velocity components (*u, v, w*) yields a mathematical formulation of Taylor's hypothesis

$$\frac{\partial \zeta}{\partial t} = -\vec{v} \cdot \nabla \zeta \,, \tag{3.14}$$

where ∇ denotes the gradient operator. Thus, if the turbulence is 'frozen' the local temporal change in any variable ξ manifests itself through the advection of an eddy across the sensor. In other words, Eq. (3.14) suggests a simple transformation of variables characterising the turbulence in time (V_t) to those characterising it in space (V_{x_i}) using the mean wind speed: $(V_x, V_y, V_z) = (\overline{u}, \overline{v}, \overline{w}) \cdot V_t$.

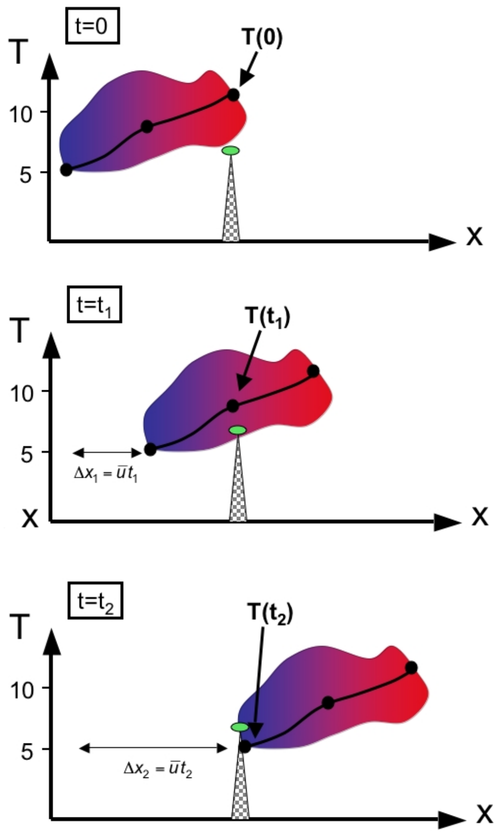

Figure 3.4 Illustration of Taylor's Hypothesis (adapted after Stull 1988).

As an example we may consider the information one can extract from turbulence spectra (Chapter 7). From a measured time series the spectral distribution of energy can be obtained (see, e.g., Fig. 7.5) and hence f_{max}, the frequency of maximum spectral power. However, often the corresponding wave length, λ_{max}, is of interest because it is related to the *size of the dominant eddies*. Taylor's hypothesis then yields for the one-dimensional spectrum in mean wind direction

$$\lambda_{max} = \overline{u} / f_{max}, \qquad (3.15)$$

having noted that the frequency is inversely related to time.

Under what circumstance is Taylor's hypothesis applicable in real boundary layer flows? In general, the characteristic time over which turbulent eddies change due to the forcing, T_f, should be much larger than the travel 'across the sensor', or

$$T_f >> L_e / \overline{u}\,. \tag{3.16}$$

Here, L_e is a characteristic length scale of an eddy. In practice, from measurements a simple rule-of-the-thumb for the validity of Taylor's hypothesis has been devised as

$$\sigma_u / \overline{u} < 0.5\,, \tag{3.17}$$

where σ_u (i.e., the standard deviation) represents the activity of turbulence and $\overline{u}$ the mean advection velocity.

3.3 Reynolds Decomposition

In the previous section the averaging operator has been introduced without explicitly stating which part of the variability is considered as turbulence and which as 'mean flow'. In other words, we have not dealt with the question of averaging times on the one hand, and neither with the separation of turbulence and mean flow, on the other hand. Inspecting Fig. 3.1 for example leads to the question of how to separate these different processes in a real time series. Osborne Reynolds has introduced a simple approach that is ever since referred to as *Reynolds Decomposition* to address the second question. In this approach every variable *a* is formally separated in to a mean ($\overline{a}$) and a fluctuating (turbulent) part (*a'*) according to

$$a = \overline{a} + a'\,. \tag{3.18}$$

(Note that this formal decomposition does not alter the information in the signal in any way). However, the Reynolds Decomposition is only useful if also the first question can be answered satisfactorily. Here, the nature of the energy distribution in many natural flows helps to find an answer. Figure 3.5 shows the energy spectrum[4] as recorded over many days by a turbulence probe. Clearly, it can be seen that a local minimum in energy is present between local maxima at both the high and low frequency ends of the spectrum. This minimum is referred to as the *spectral gap*. On the low frequency end the variability due to the daily cycle and longer-term developments (e.g., changing weather conditions) dominate the spectrum, while the high-frequency variability beyond the spectral gap can be identified with turbulence. The presence of a spectral gap thus allows the separation of scales ('mean flow' vs. 'turbulence') and the application of (3.18). Often, it occurs at a frequency corresponding to a period of some, say, fifteen minutes to a couple of hours. In practice, therefore, one measures (or models) a high-frequency time series, of *a* and then calculates $\overline{a}$ by choosing an appropriate averaging time. The turbulent fluctuations (*a'*) can then be determined by (3.18).

Table 3.2 compiles the calculus rules for the Reynolds Decomposition for the example of two variables *a* and *b*. While most of these rules are obvious it is instructive to note that the mean product of the two variables is not equal to the simple product of the two mean variables. In general, the average of the

[4] See Chapter 7 for details.

product of the two fluctuating parts $\overline{a'b'}$ does not vanish thus indicating that the two fluctuations are correlated. In turbulent flows, and especially close to the ground, these *covariances* are often larger in magnitude than the corresponding product of the mean variables and thus dominate the flow.

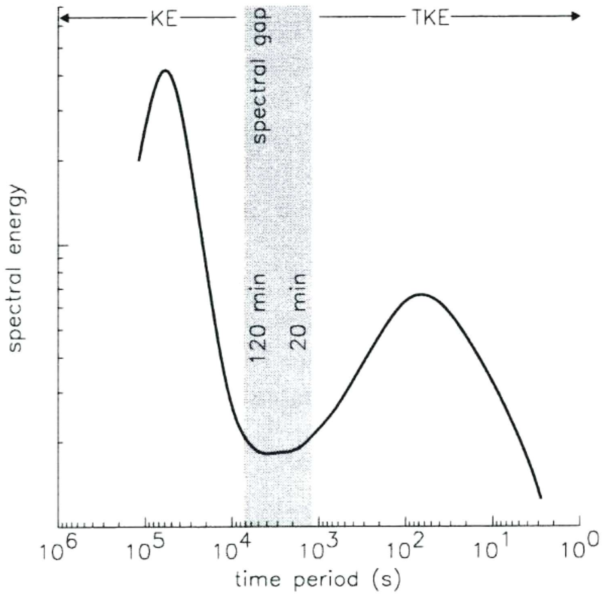

Figure 3.5 Illustration of spectral gap – mean spectral density distribution over a seven hours period at a site in a Alpine valley at z/h=1.74 over a forest stand. From van Gorsel (2003).

3.4 Covariances and their Physical Meaning

Transport in a physical system can most generally be described as the product of a transport velocity with the transported quantity. In atmospheric (fluid dynamical) systems this is usually called advection. When introducing the concept of Reynolds Decomposition to the conservation equations in Chapter 5 we will face the fact that the advection terms (e.g., $u_i \partial\theta_j / \partial x_j$ [5] in the energy conservation equation, where θ denotes the potential temperature) give rise to additional covariance terms as those in Table 3.2. These can be interpreted as describing *turbulent transport* (or flux) terms as outlined in the following.

To understand transport in general, consider a tube as sketched in Fig. 3.6 with length Δx, volume V and the head area ΔA. Let χ be a specific quantity, such as specific humidity. We then may define the integral quantity according to $\mathrm{X} = \iiint_V \rho\chi dV$. If χ is well-mixed within the volume the transport through area ΔA becomes

[5] Note that the advection term is only non-zero if the transported quantity changes in flow direction. In the given example, if θ does not change in j-direction, there *is transport of sensible heat* in this direction, but it does not change the temperature (because air of a certain temperature is transported into an area with the same temperature).

$$F_X = \frac{X}{\Delta A \Delta t} = \frac{\rho\chi V}{\Delta A \Delta t} = \frac{\rho\chi \Delta A \Delta x}{\Delta A \Delta t} = \rho\chi \frac{\Delta x}{\Delta t} \tag{3.19}$$

Table 3.2: Calculus for Reynolds Decomposition

	a and b are variables, for which: $a = \bar{a} + a'$; $b = \bar{b} + b'$	
1)	$\overline{a'} = 0$	By definition
2)	$\overline{(a)} = \overline{(\bar{a} + a')} = \bar{a}$	By definition and 1)
3)	$\overline{(\bar{b} \cdot a')} = \bar{b} \cdot \overline{a'} = 0$	The average of a product involving a primed variable vanishes
4)	$\overline{(a \cdot b)} = \overline{(\bar{a} + a') \cdot (\bar{b} + b')}$ $= \bar{a} \cdot \bar{b} + \overline{a'b'}$	The covariance is not necessarily zero
5)	$\overline{a^2} = \bar{a}^2 + \overline{a'^2}$	The second term on the rhs corresponds to the Second central moment, i.e. the variance

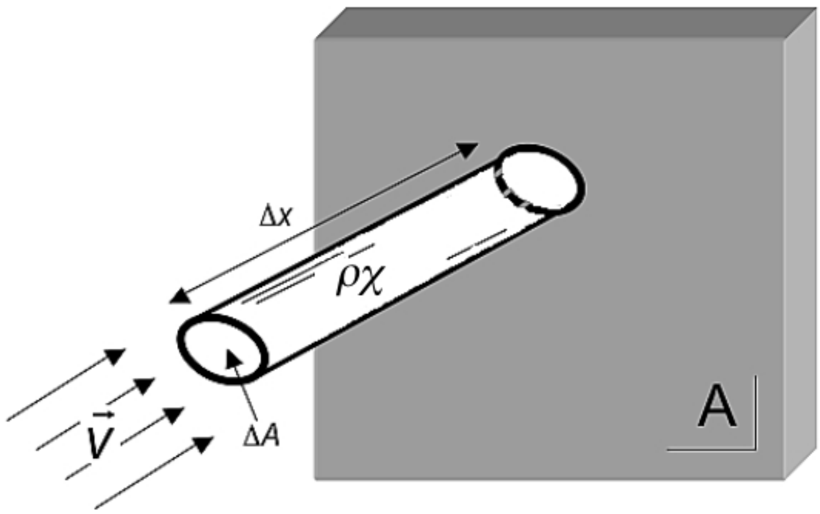

Figure 3.6 Illustration for the concept of transport in a fluid.

Thus in the limit of infinitesimal increments (and expanding to three dimensions) a flux of a quantity χ can most generally be described as

$$\vec{F}_\chi = \rho\chi\vec{v}\,. \tag{3.20}$$

(i.e., the product of the fluid's density with the specific quantity and the transport velocity). The most important covariances that appear when applying the Reynolds Decomposition into the conservation equations in Chapter 5 are always associated with the vertical velocity component, viz.

$$\overline{w'\theta'},\ \overline{w'q'} \text{ and } \overline{u'w'}, \tag{3.21}$$

i.e., the turbulent fluxes of sensible heat, latent heat (q denoting the specific humidity) and momentum, respectively. Thus, if for example the covariance between the (fluctuating) vertical velocity component and the (fluctuating) potential temperature does not vanish a turbulent flow can transport sensible heat in the vertical *even without a mean vertical velocity*. Experience shows that indeed this is the case especially close to the surface (or more generally, within the PBL) so that the terms including the covariances of Eq. (3.21) even dominate the respective conservation equations.

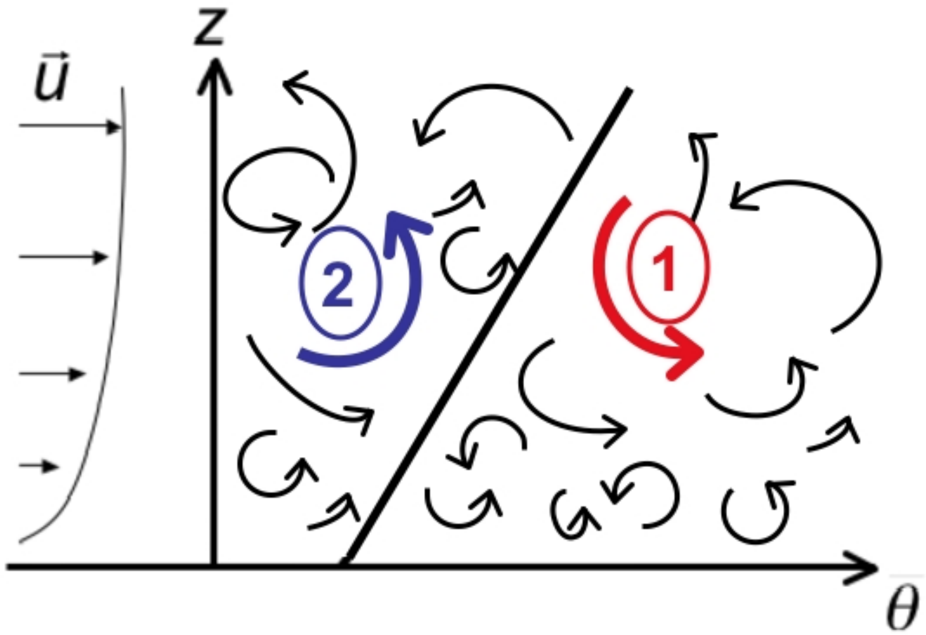

Figure 3.7 Sketch of a turbulent flow with the shear on the left producing turbulence. Two 'eddies' ('1' and '2') are outlined.

Consider a turbulent flow as sketched in Fig. 3.7. The turbulence is symbolised as curved velocity vectors denoting instantaneous turbulent eddies of different size. The mean temperature stratification is also indicated. Turbulent eddy labelled 1 is an example of a downward fluctuation, and due to the mean temperature stratification it is *likely* to transport air that is relatively warm into a region that is colder. On the other hand eddy 2 represents a typical upward fluctuation and brings relatively cool air into a region of larger potential temperature. Eddy 1 transports heat downwards, while eddy 2 transports heat deficit upwards. In our coordinate system with the positive z-axis pointing upward both eddies contribute with a negative instantaneous flux of sensible heat. Averaged over many such eddies (i.e., a certain averaging period) the covariance between w' and θ' is negative (*in this example!*) or, in other words, heat is transported towards the surface through turbulent transport. This is illustrated in Fig. 3.8. In general, every combination is possible between the fluctuations of w and θ in our example leading to the four 'quadrants' of Fig. 3.9. Only if – due, for example, to the mean gradient in one of the variables as in the example – either one of the diagonals dominates the other a non-zero average covariance results leading to a net transport of the considered scalar.

So far, we have only considered the covariances alone (Eq. 3.20). These are called *kinematic fluxes*. They are converted into physical units through

$\rho c_p \overline{w'\theta'} =: H =$ turbulent flux of sensible heat in Wm^{-2} (c_p = specific heat of air at constant pressure);

$\rho L_v \overline{w'q'} =: L_v E =$ turbulent flux of latent heat in Wm^{-2} (L_v = latent heat of condensation);

$\rho \overline{u'w'} =: M =$ turbulent flux of momentum in Nm^{-2}.

The momentum flux is special in as the transported 'scalar' is in fact also a velocity component. When applying the concept of Reynolds Decomposition and averaging to the product of two velocities (as it appears in the conservation equation for momentum) one in fact obtains a tensor

$$\overline{\mathrm{cov}(u_i, u_j)} = \overline{u_i' u_j'} = \begin{Bmatrix} \overline{u'u'} & \overline{u'v'} & \overline{u'w'} \\ \overline{v'u'} & \overline{v'v'} & \overline{v'w'} \\ \overline{w'u'} & \overline{w'v'} & \overline{w'w'} \end{Bmatrix} \tag{3.22}$$

It can easily be seen that the diagonal elements correspond to the velocity variances.

What was called 'momentum flux' above thus corresponds only to one element of this tensor. Since both fluctuating variables in this tensor element are velocity components one is tempted to ask: does this now correspond to turbulent transport of horizontal momentum in the vertical or rather to turbulent transport of vertical momentum in the horizontal (along wind) direction? To answer this question it is instructive to consider the physical effect these two covariances represent on an idealised 'fluid element' as sketched in Fig. 3.10. If an 'upward fluctuation' ($w' > 0$) is transported towards the side of the fluid element (Fig. 3.10a) the result is a deformation of the original fluid element (Fig. 3.10b). Similarly, if a gust ($u' > 0$) is transported towards the fluid element from, say, above (Fig. 3.10c) a deformation results (Fig. 3.10d). Both these deformations are equal in principle and correspond to a stretching of the fluid element in the x-z plane (Fig. 3.10e). Thus $\overline{u'w'}$ and $\overline{w'u'}$ are equal in their effect and express the *shear stress* the fluid element experiences and hence the tensor in (3.22) is symmetric (only 6 independent elements – with the same arguments for the other off-diagonal element pairs).

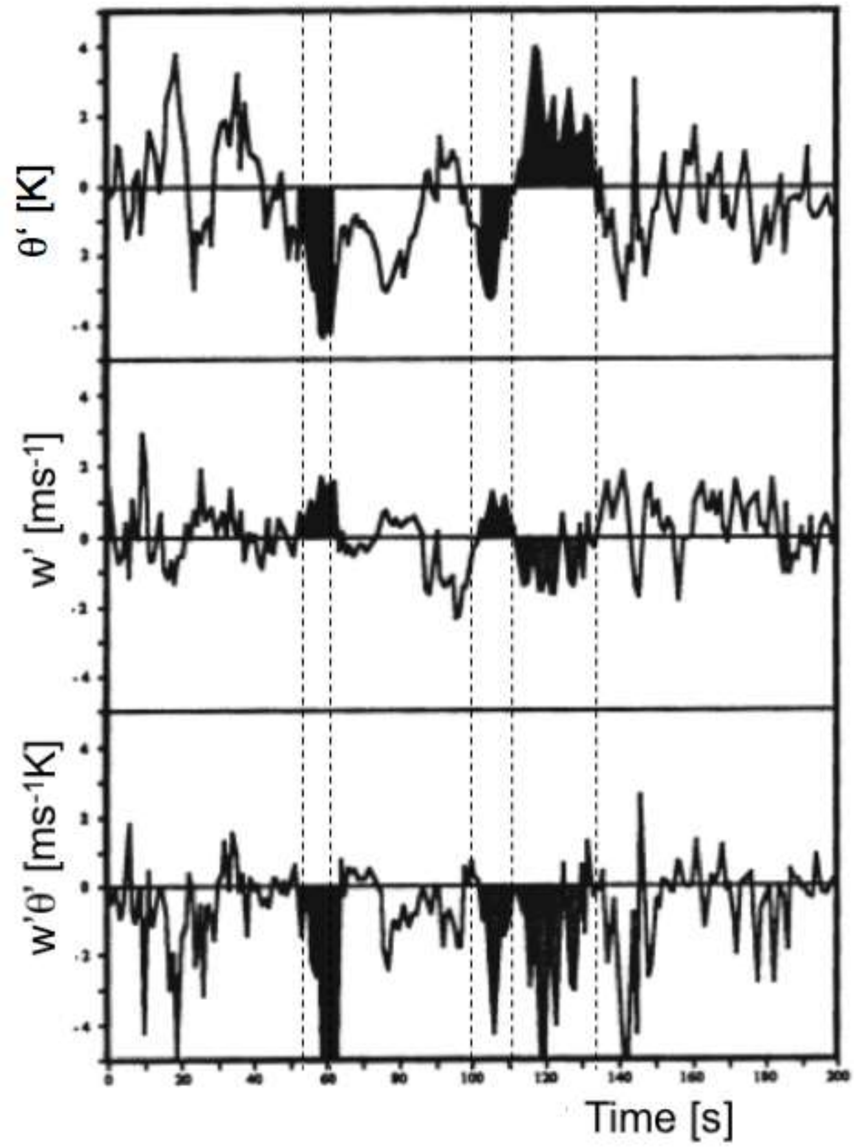

Figure 3.8 Time series of θ' (upper panel), w' (middle panel) and the (instantaneous) $w'\theta'$ (lower panel). Dashed vertical lines (and dark shading) connect periods of interest. Adapted from Rotach (1993).

A *Reynolds stress tensor* (or shear stress tensor) is therefore defined according to

$$\tau_{ij} \equiv -\rho \overline{u_i' u_j'}. \tag{3.23}$$

The minus sign is thereby introduced in order to assure that the most important element (τ_{13}) is positive: due to friction the profile of mean wind speed close to the surface always obeys $\partial \overline{u} / \partial z > 0$ (vanishing mean wind speed *at* the surface) and hence $\overline{u'w'} < 0$ (cf. the example of Fig. 3.7).

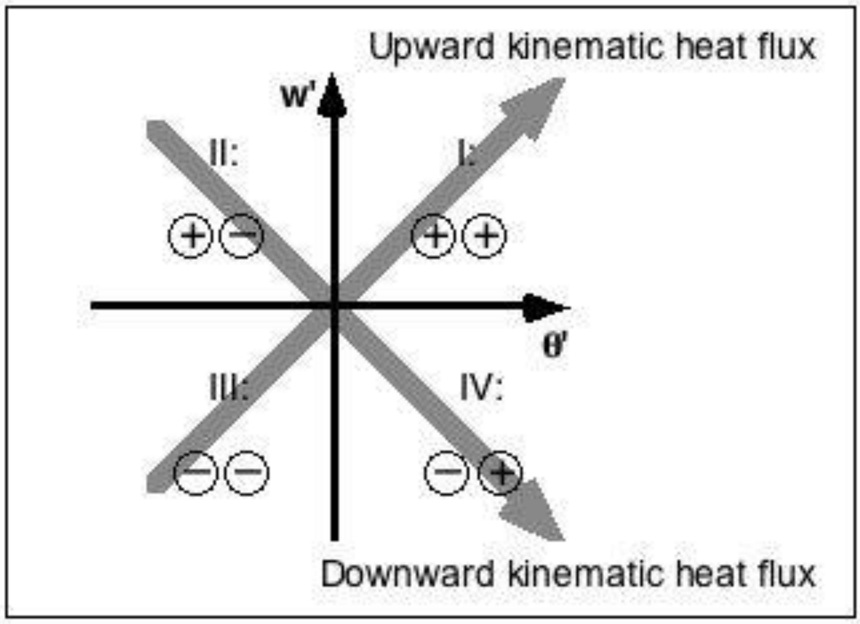

Figure 3.9 Illustration of the four quadrants with the ‘plus’ and ‘minus’ signs denoting the potential temperature and vertical velocity fluctuations respectively.

In summary, the surface friction leads to a permanent deformation (shear stress) of the fluid elements close to the surface and the mean velocity profile is maintained through turbulent transport of momentum towards the surface. Due to the importance of this processes in the generation of mechanical turbulence a characteristic velocity is defined according to

$$u_* \equiv \left[(\overline{u'w'_o})^2 + (\overline{v'w'_o})^2 \right]^{1/4} \tag{3.24}$$

and called *friction velocity*. Here, the subscript ‘o’ denotes the surface value.

3.5 Other Turbulence Variables

The three diagonal elements in (3.22) have been identified as the velocity variances above. They in fact represent the normal stresses on the ‘fluid element’ of Fig. 3.10. Clearly, they could be interpreted as ‘transport of u_i in direction x_i. More important, however is their meaning in terms of energy. The kinetic energy is most generally defined as $E_{kin} = \frac{\rho}{2}(u^2 + v^2 + w^2)$. Applying the Reynolds Decomposition and averaging then leads to the *turbulent kinetic energy* (TKE):

$$TKE = \frac{1}{2}\rho\overline{u_{ii}'^2}\ . \tag{3.25}$$

Also from the velocity variances the *turbulent intensities* are defined

$$I_k = \frac{\sigma_{u_k}}{\overline{u}}, \tag{3.26}$$

where σ_{u_k} is the standard deviation of the respective velocity component.

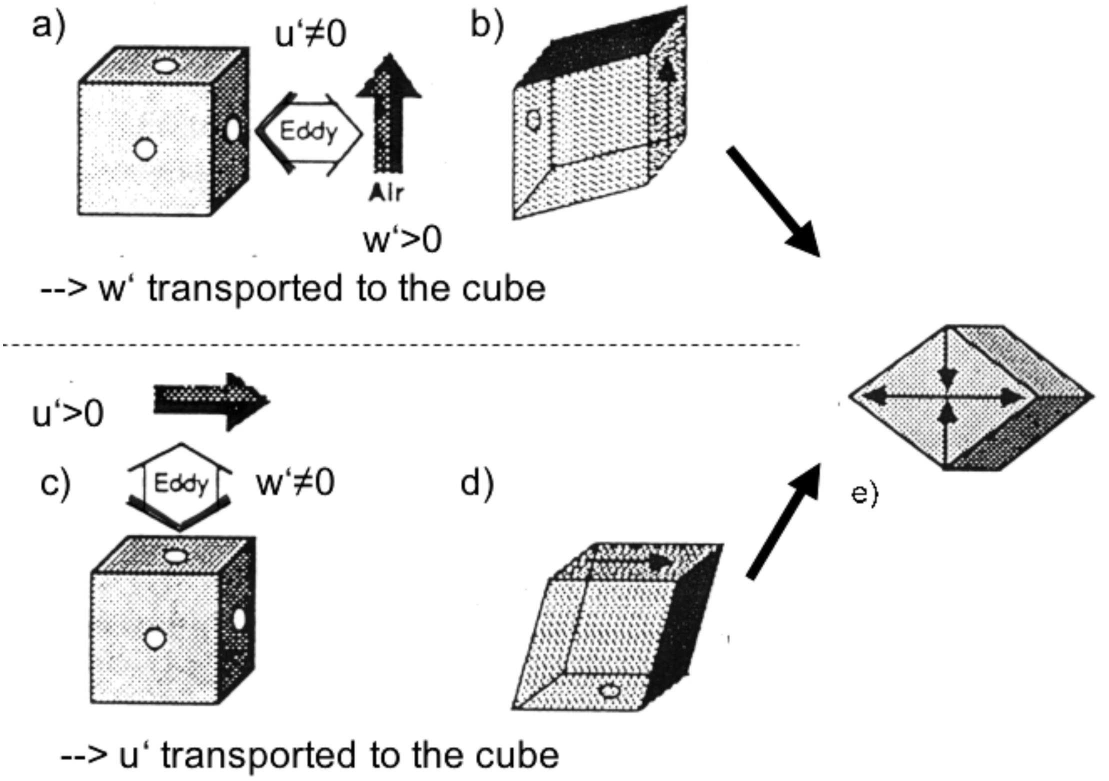

Figure 3.10 Cubes illustrating horizontal transport of vertical velocity fluctuation (a) and resulting deformation (b); transport of vertical velocity fluctuation (c) and resulting deformation (d); equivalence of b) and d) in e). Modified after Stull (1988)

The *auto correlation function* R_a of a variable a describes the correlation of this variable with itself as a function of time difference or distance in space. For example, the temporal auto-correlation function is defined

$$R_a(t,\tau) =: \frac{\overline{a'(t)\cdot a'(t+\tau)}}{\overline{a'^2}(t)}. \tag{3.27}$$

Here, t denotes the time and τ the time difference. From this definition it is clear that $R_a(t,0) = 1$. Furthermore in stationary turbulence R_a is also independent of t. In general[6], R_a decreases with increasing τ (Fig. 3.11). From (3.27) the so-called integral time scale for variable a is defined after

$$T_a(\tau) \equiv \int_0^{\infty} R_a(\tau) d\tau . \tag{3.28}$$

Figure 3.11 shows that the integral time scale is measure for the 'memory' of the turbulence: the lager T_a the longer at least some correlation is maintained for the considered variable.

[6] If we consider turbulence time scales – and exclude periodic phenomena (such as waves)

An often-used model approach for the auto-correlation function reads

$$R_a(\tau) = \exp\left\{-\tau / T_a\right\}. \tag{3.29}$$

From (3.29) it is immediately clear that the time difference $\tau = T_a$ corresponds to the e-folding time (difference) for variable a. Inserting (3.29) into (3.28) shows that this model approach for R_a is consistent with the definition of the integral time scale (i.e., the integration of (3.29) yields T_a).

In a similar manner the auto correlation function in space and a corresponding integral length scale can be defined (in one spatial dimension for simplicity):

$$R_{a,x}(x,\Delta x) \equiv \frac{\overline{a'(x) \cdot a'(x+\Delta x)}}{\overline{a'^2}(x)}. \tag{3.30}$$

and

$$L_{a,x}(\Delta x) \equiv \int_0^\infty R_{a,x}(\Delta x) d\Delta x. \tag{3.31}$$

These integral scales will play an important role in developing a theory for the dispersion of air pollutant in a turbulent flow, the so-called Taylor theory (Chapter 10).

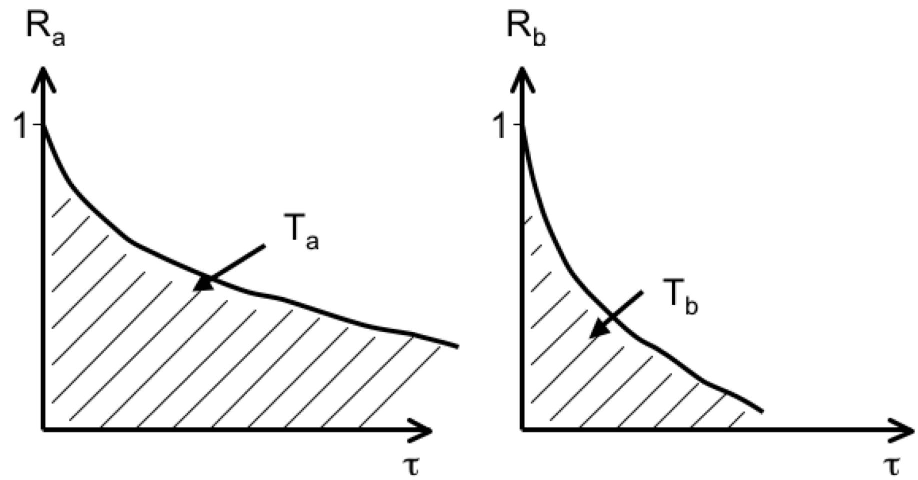

Figure 3.11 Auto correlation function for a variable with large (left) and small (right) Integral time scale.

References Chapter 3

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